Zeilberger's Algorithm

How the algorithm works

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Introduction

Overall goal:

We want to find a closed form of $f(n)$ for given $f(n) = \sum_{k=0}^{\infty} F(n,k)$.

Exact form of the recurrence formula:
$\sum_{j=0}^{J} a_j(n) F(n+j,k) = G(n,k+1) - G(n,k)$

First part of the algorithm: reshaping and rearranging terms

1: reshaping and rearranging terms

Assume order of recurrence J and fix it.

Recurrence formula: $\sum_{j=0}^{J} a_j(n) F(n+j,k) = G(n,k+1) - G(n,k)$

$t_k := \sum_{j=0}^{J} a_j(n) F(n+j,k)$

$$\frac{t_{k+1}}{t_k} = \frac{\sum_{j=0}^{J} a_j(n) F(n+j,k+1)/F(n,k+1)}{\sum_{j=0}^{J} a_j(n) F(n+j,k)/F(n,k)} \cdot \frac{F(n,k+1)}{F(n,k)}$$

We define $\frac{r_1(n,k)}{r_2(n,k)} := \frac{F(n,k+1)}{F(n,k)}$ and $\frac{s_1(n,k)}{s_2(n,k)} := \frac{F(n,k)}{F(n-1,k)}$.

$$\frac{F(n+j,k)}{F(n,k)} = \prod_{i=0}^{j-1} \frac{F(n+j-i,k)}{F(n+j-i-1,k)} = \prod_{i=0}^{j-1} \frac{s_1(n+j-i,k)}{s_2(n+j-i,k)}$$

1: reshaping and rearranging terms

Substituting $\frac{r_1(n,k)}{r_2(n,k)} := \frac{F(n,k+1)}{F(n,k)}$ and

$\frac{F(n+j,k)}{F(n,k)} = \prod_{i=0}^{j-1} \frac{F(n+j-i,k)}{F(n+j-i-1,k)} = \prod_{i=0}^{j-1} \frac{s_1(n+j-i,k)}{s_2(n+j-i,k)}$

into $\frac{t_{k+1}}{t_k} = \frac{\sum_{j=0}^{J} a_j(n)F(n+j,k+1)/F(n,k+1)}{\sum_{j=0}^{J} a_j(n)F(n+j,k)/F(n,k)} \cdot \frac{F(n,k+1)}{F(n,k)}$:

We get $\frac{t_{k+1}}{t_k} = \frac{\sum_{j=0}^{J} a_j \left(\prod_{i=0}^{j-1} \frac{s_1(n+j-i,k+1)}{s_2(n+j-i,k+1)}\right) \cdot r_1(n,k)}{\sum_{j=0}^{J} a_j \left(\prod_{i=0}^{j-1} \frac{s_1(n+j-i,k)}{s_2(n+j-i,k)}\right) \cdot r_2(n,k)}$.

1: reshaping and rearranging terms

goal: expression of the form $\frac{t_{k+1}}{t_k} = \frac{p_0(k+1)}{p_0(k)} \frac{r(k)}{s(k)}$

$$\frac{t_{k+1}}{t_k} = \frac{\sum_{j=0}^{J} a_j \left(\prod_{i=0}^{j-1} \frac{s_1(n+j-i,k+1)}{s_2(n+j-i,k+1)}\right) \cdot r_1(n,k)}{\sum_{j=0}^{J} a_j \left(\prod_{i=0}^{j-1} \frac{s_1(n+j-i,k)}{s_2(n+j-i,k)}\right) \cdot r_2(n,k)} \text{ is equal to}$$

$$\frac{\sum_{j=0}^{J} a_j \left(\prod_{i=0}^{j-1} s_1(n+j-i,k+1) \prod_{r=j+1}^{J} s_2(n+r,k+1)\right)}{\sum_{j=0}^{J} a_j \left(\prod_{i=0}^{j-1} s_1(n+j-i,k) \prod_{r=j+1}^{J} s_2(n+r,k)\right)} \cdot \frac{r_1(n,k) \prod_{r=1}^{J} s_2(n+r,k)}{r_2(n,k) \prod_{r=1}^{J} s_2(n+r,k+1)}.$$

1: reshaping and rearranging terms

$$\frac{t_{k+1}}{t_k} = \frac{\sum_{j=0}^{J} a_j \left(\prod_{i=0}^{j-1} s_1(n+j-i,k+1) \prod_{r=j+1}^{J} s_2(n+r,k+1)\right)}{\sum_{j=0}^{J} a_j \left(\prod_{i=0}^{j-1} s_1(n+j-i,k) \prod_{r=j+1}^{J} s_2(n+r,k)\right)} \cdot \frac{r_1(n,k) \prod_{r=1}^{J} s_2(n+r,k)}{r_2(n,k) \prod_{r=1}^{J} s_2(n+r,k+1)}$$

is now an expression of the form $\frac{t_{k+1}}{t_k} = \frac{p_0(k+1)}{p_0(k)} \frac{r(k)}{s(k)}$.

Where: $p_0\left(k\right) = \sum_{j=0}^{J} a_j \left(\prod_{i=0}^{j-1} s_1\left(n+j-i,k\right) \prod_{r=j+1}^{J} s_2\left(n+r,k\right)\right)$,

$r\left(k\right) = r_1\left(n,k\right) \prod_{r=1}^{J} s_2\left(n+r,k\right)$,

$s\left(k\right) = r_2\left(n,k\right) \prod_{r=1}^{J} s_2\left(n+r,k+1\right)$.

Remark: $\{a_j\}_{j=0}^{J}$ just appear in $p_0\left(k\right)$.

Example

1: example

$f(n) = \sum_{k=0}^{\infty} F(n,k)$ with $F(n,k) = \binom{2n}{2k}$

$\sum_{j=0}^{J} a_j(n) F(n+j,k) = G(n,k+1) - G(n,k)$

Assume J=1.

1: example

$t_k := \sum_{j=0}^{J} a_j(n)\, F(n+j,k)$ and $F(n,k) = \binom{2n}{2k}$

$\frac{t_{k+1}}{t_k} = \frac{a_0\binom{2n}{2k+2}+a_1\binom{2n+2}{2k+2}}{a_0\binom{2n}{2k}+a_1\binom{2n+2}{2k}}$

Goal: bring this in the form $\frac{t_{k+1}}{t_k} = \frac{p_0(k+1)}{p_0(k)}\frac{r(k)}{s(k)}$

$$\frac{t_{k+1}}{t_k} = \frac{a_0\binom{2n}{2k+2}+a_1\binom{2n+2}{2k+2}}{a_0\binom{2n}{2k}+a_1\binom{2n+2}{2k}} = \frac{a_0\frac{2n!}{(2k+2)!(2n-2k-2)!}+a_1\frac{(2n+2)!}{(2k+2)!(2n-2k)!}}{a_0\frac{2n!}{(2k)!(2n-2k)!}+a_1\frac{(2n+2)!}{2k!(2n-2k+2)!}}$$

$$= \frac{a_0\frac{1}{(2k+1)(2k+2)}+a_1\frac{(2n+1)(2n+2)}{(2k+1)(2k+2)(2n-2k-1)(2n-2k)}}{a_0\frac{1}{(2n-2k-1)(2n-2k)}+a_1\frac{(2n+1)(2n+2)}{(2n-2k-1)(2n-2k)(2n-2k+1)(2n-2k+2)}}$$

1: example

$$\frac{a_0 \frac{1}{(2k+1)(2k+2)} + a_1 \frac{(2n+1)(2n+2)}{(2k+1)(2k+2)(2n-2k-1)(2n-2k)}}{a_0 \frac{1}{(2n-2k-1)(2n-2k)} + a_1 \frac{(2n+1)(2n+2)}{(2n-2k-1)(2n-2k)(2n-2k+1)(2n-2k+2)}}$$

Expand with $(2n-2k-1)(2n-2k)$:

$$= \frac{a_0 \frac{(2n-2k-1)(2n-2k)}{(2k+1)(2k+2)} + a_1 \frac{(2n+1)(2n+2)}{(2k+1)(2k+2)}}{a_0 + a_1 \frac{(2n+1)(2n+2)}{(2n-2k+1)(2n-2k+2)}}$$

$$= \frac{a_0(2n-2k-1)(2n-2k)+a_1(2n+1)(2n+2)}{a_0(2n-2k+1)(2n-2k+2)+a_1(2n+1)(2n+2)} \cdot \frac{(2n-2k+1)(2n-2k+2)}{(2k+1)(2k+2)}$$

1: example

$$\frac{t_{k+1}}{t_k} = \frac{a_0(2n-2k-1)(2n-2k)+a_1(2n+1)(2n+2)}{a_0(2n-2k+1)(2n-2k+2)+a_1(2n+1)(2n+2)} \cdot \frac{(2n-2k+1)(2n-2k+2)}{(2k+1)(2k+2)}$$

$$\frac{t_{k+1}}{t_k} = \frac{p_0\left(k+1\right)}{p_0\left(k\right)}\frac{r\left(k\right)}{s\left(k\right)}$$

So it holds

$p_0\left(k\right) = a_0\left(2n-2k+1\right)\left(2n-2k+2\right) + a_1\left(2n+1\right)\left(2n+2\right),$

$r\left(k\right) = \left(2n-2k+1\right)\left(2n-2k+2\right),$

$s\left(k\right) = \left(2k+1\right)\left(2k+2\right).$

Second part of the algorithm: use of the previous chapter about Gosper's algorithm

2: use of the previous chapter about Gosper's algorithm

Theorem 5.3.1

Let K be a field of characteristic zero and $r \in K[n]$ a nonzero rational function. Then there exist polynomials $a, b, c \in K[n]$ such that b, c are monic and $r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}$, where

1. $gcd(a(n), b(n+k)) = 1$ *for every nonnegative integer k,*
2. $gcd(a(n), c(n)) = 1$,
3. $gcd(b(n), c(n+1)) = 1$.

So we write $\frac{r(k)}{s(k)}$ as $\frac{r(k)}{s(k)} = \frac{p_1(k+1)}{p_1(k)} \cdot \frac{p_2(k)}{p_3(k)}$.

$\frac{t_{k+1}}{t_k} = \frac{p_0(k+1)}{p_0(k)} \frac{r(k)}{s(k)}$

2: use of the previous chapter about Gosper's algorithm

Now we define $p(k)$: $p(k) = p_0(k)\, p_1(k)$

p_0 was defined earlier as follows:

$\frac{t_{k+1}}{t_k} = \frac{p_0(k+1)}{p_0(k)} \frac{r(k)}{s(k)}$ with

$p_0(k) = \sum_{j=0}^{J} a_j \left(\prod_{i=0}^{j-1} s_1(n+j-i,k) \prod_{r=j+1}^{J} s_2(n+r,k) \right)$

By the definition of $p(k)$, $\frac{t_{k+1}}{t_k} = \frac{p_0(k+1)}{p_0(k)} \frac{r(k)}{s(k)}$ and $\frac{r(k)}{s(k)} = \frac{p_1(k+1)}{p_1(k)} \cdot \frac{p_2(k)}{p_3(k)}$ we get $\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \cdot \frac{p_2(k)}{p_3(k)}$.

Remark: Coefficients of $p_2(k)$ and $p_3(k)$ are independent of $\{a_j\}_{j=0}^{J}$ and $p(k)$ depends on $\{a_j\}_{j=0}^{J}$ only linearly.

2: use of the previous chapter about Gosper's algorithm

$\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \cdot \frac{p_2(k)}{p_3(k)}$ setup for Gosper's algorithm

t_k is an indefinitely summable hypergeometric term if and only if the recurrence formula $p_2(k)\, b(k+1) - p_3(k-1)\, b(k) = p(k)$ has a polynomial solution $b(k)$.

Remark: A hypergeometric term t_k is indefinitely summable if $\sum_{k=0}^{a} t_k$ has a closed form, where a is an arbitrary upper bound.

Find an upper bound on the degree of the polynomial $b(k)$ (page 85, chapter 5).

2: use of the previous chapter about Gosper's algorithm

$z_n = \sum_{k=0}^{n-1} e_k$, $r(k) = \frac{e_{k+1}}{e_k}$

goal: z_n such that $z_{n+1} - z_n = e_n$ (1)

$z(n) := y(n)e_n$, $y(n)$ unknown rational function

substituting $y(n)e_n$ for z_n in (1), we get: $r(n)y(n+1) - y(n) = 1$ (2)

$r(n) = \frac{a(n)}{b(n)}\frac{c(n+1)}{c(n)}$ (3)

we look for $y(n)$ in the form $y(n) = \frac{b(n-1)x(n)}{c(n)}$, $x(n)$ (4) rational function of n

substituting (3) and (4) into (2) leads to
$a(n)x(n+1) - b(n-1)x(n) = c(n)$ (5), theorem 5.2.1: $x(n)$ is a polynomial

finding hypergeometric solutions of $z_{n+1} - z_n = t_n$ equivalent to finding polynomial solution of (5), if $x(n)$ is a nonzero polynomial solution of (5), then $z_n = \frac{b(n-1)x(n)}{c(n)}e_n$

2: example

part 1: $\frac{t_{k+1}}{t_k} = \frac{p_0(k+1)}{p_0(k)} \frac{r(k)}{s(k)}$

part 2: find the canonical form $\frac{r(k)}{s(k)} = \frac{p_1(k+1)}{p_1(k)} \cdot \frac{p_2(k)}{p_3(k)}$

$r\left(k\right) = \left(2n - 2k + 1\right)\left(2n - 2k + 2\right) = 4n^2 + 4k^2 + 2 - 8kn + 6n - 6k$

$s\left(k\right) = \left(2k + 1\right)\left(2k + 2\right) = 4k^2 + 6k + 2$

$$\frac{r\left(k\right)}{s\left(k\right)} = \frac{1}{1} \cdot \frac{2n^2 + 2k^2 + 1 - 4kn + 3n - 3k}{2k^2 + 3k + 1}$$

So we have $p_1\left(k\right) = 1$, $p_1\left(k + 1\right) = 1$,

$p_2\left(k\right) = 2n^2 + 2k^2 + 1 - 4kn + 3n - 3k$,

$p_3\left(k\right) = 2k^2 + 3k + 1$.

2: example

We write $\frac{t_{k+1}}{t_k}$ as

$$\frac{t_{k+1}}{t_k} = \frac{p\left(k+1\right)}{p\left(k\right)} \cdot \frac{p_2\left(k\right)}{p_3\left(k\right)}$$

$$\left(p\left(k\right) = p_0\left(k\right)p_1\left(k\right) = a_0\left(2n-2k+1\right)\left(2n-2k+2\right)+a_1\left(2n+1\right)\left(2n+2\right)\right)$$

$$\frac{t_{k+1}}{t_k} = \frac{a_0(2n-2k-1)(2n-2k)+a_1(2n+1)(2n+2)}{a_0(2n-2k+1)(2n-2k+2)+a_1(2n+1)(2n+2)} \cdot \frac{2n^2+2k^2+1-4kn+3n-3k}{2k^2+3k+1}$$

2: example

$$\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \cdot \frac{p_2(k)}{p_3(k)}$$

$$\frac{t_{k+1}}{t_k} = \frac{a_0(2n-2k-1)(2n-2k)+a_1(2n+1)(2n+2)}{a_0(2n-2k+1)(2n-2k+2)+a_1(2n+1)(2n+2)} \cdot \frac{2n^2+2k^2+1-4kn+3n-3k}{2k^2+3k+1}$$

$$p_2(k)\, b(k+1) - p_3(k-1)\, b(k) = p(k)$$

$$\left(2n^2+2k^2+1-4kn+3n-3k\right) b(k+1) - \left(2k^2-k\right) b(k) =$$

$$a_0 (2n-2k+1)(2n-2k+2) + a_1 (2n+1)(2n+2)$$

By page 85 in chapter 5 of the book we find out that $b(k)$ has degree 1, we write $b(k)$ as $b(k) = c + dk$.

Third part of the algorithm: solve system of linear equations

3: solve system of linear equations

A: upper bound for $b(k)$, $b(k) = \sum_{l=0}^{A} c_l k^l$.

Substitute this into $p_2(k)\, b(k+1) - p_3(k-1)\, b(k) = p(k)$.

We get a system of linear equations (by doing coefficient comparison) with the unknown coefficients $\{a_j\}_{j=0}^{J}$ and $\{c_l\}_{l=0}^{A}$.

Solve the system, we get the coefficients $\{a_j\}_{j=0}^{J}$ of the recurrence formula $\sum_{j=0}^{J} a_j(n)\, F(n+j, k) = G(n, k+1) - G(n, k)$.

3: solve system of linear equations

Additionally, by equation (5.2.5) in chapter 5.2 we get G(n,k):

Equation 5.2.5: $y\left(n\right)=\frac{b(n-1)x(n)}{c(n)}$.

Therefore we can conclude $G\left(n,k\right)=\frac{p_3(k-1)}{p(k)}b\left(k\right)t_k$.

Remark: The system of linear equations has usually not a unique solution but the solution is determined up to a constant.

$\sum_{j=0}^{J} a_j\left(n\right)F\left(n+j,k\right)=G\left(n,k+1\right)-G\left(n,k\right)$

2: use of the previous chapter about Gosper's algorithm

$z_n = \sum_{k=0}^{n-1} e_k$, $r(k) = \frac{e_{k+1}}{e_k}$

goal: z_n such that $z_{n+1} - z_n = e_n$ (1)

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$r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}$ (3)

we look for $y(n)$ in the form $y(n) = \frac{b(n-1)x(n)}{c(n)}$, $x(n)$ (4) rational function of n

substituting (3) and (4) into (2) leads to $a(n)x(n+1) - b(n-1)x(n) = c(n)$ (5), theorem 5.2.1: $x(n)$ is a polynomial

finding hypergeometric solutions of $z_{n+1} - z_n = t_n$ equivalent to finding polynomial solution of (5), if $x(n)$ is a nonzero polynomial solution of (5), then $z_n = \frac{b(n-1)x(n)}{c(n)} e_n$

3: example

$$\left(2n^2 + 2k^2 + 1 - 4kn + 3n - 3k\right) b\left(k+1\right) -$$
$$\left(2k^2 - k\right) b\left(k\right) = a_0 \left(2n - 2k + 1\right)\left(2n - 2k + 2\right) + a_1 \left(2n + 1\right)\left(2n + 2\right)$$

$$b\left(k\right) = c + dk$$

By doing coefficient comparison, we can create a system of linear equations:

1. $2n^2c + c + 3nc + 2n^2d + d + 3nd =$ $4a_0n^2 + 2a_0 + 6a_0n + 4a_1n^2 + 6a_1n + 2a_1$
2. $-4nkc - 3kc + 2n^2dk + dk + 3ndk - 4ndk - 3kd + kc = -8a_0kn - 6a_0k$
3. $2k^2c - 4nk^2d - 3k^2d + 2k^2d - 2k^2c + dk^2 = 4a_0k^2$

When we solve this with maple, we get:

$$c = -3/2nd - d, a_0 = -nd, a_1 = nd/4, d = d$$

3: example

The solution is only determined up to constants. If we choose d=4, we get $a_0 = -4n$ and $a_1 = n$. This leads to the recurrence

$$nF(n+1,k) - 4nF(n,k) = G(n,k+1) - G(n,k).$$

$$\sum_{j=0}^{J} a_j(n)\, F(n+j,k) = G(n,k+1) - G(n,k)$$

Scenario A (as J=1):
$f(n) = f(1) \prod_{j=1}^{n-1} -\frac{a_0}{a_1} = 2 \prod_{j=1}^{n-1} -\frac{-4n}{n} = 2 \cdot 4^{n-1} = 2 \cdot 2^{2n-2} = 2^{2n-1}$
for $n \geq 1$.

$f(n) = \sum_{k=0}^{\infty} \binom{2n}{2k} = 2^{2n-1}$

End