Asymptotic of characters of symmetric groups and limit shape of Young diagrams

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Outline of the talk

1. Character values of symmetric groups
   - An exact formula
   - Asymptotic behaviours

2. Application: limit shape of Young diagrams
Young symmetrizer

Let $T$ be a filling of $\lambda = (3, 2, 2)$.

Consider:

row-stabilizer $RS(T) = S_{\{2,3,6\}} \times S_{\{1,4\}} \times S_{\{5,7\}}$.

column-stabilizer $CS(T) = S_{\{2,4,7\}} \times S_{\{1,3,5\}}$. 
Young symmetrizer

Let $T$ be a filling of $\lambda = (3, 2, 2)$.

\[
\frac{n! s_\lambda}{\dim \lambda} = \sum_{\substack{\sigma_1 \in RS(T) \\ \sigma_2 \in CS(T)}} (-1)^{\sigma_2} p_{\text{type}}(\sigma_2 \sigma_1)
\]

Work in progress with P. Śniady : analog for zonal polynomials
An equivalent formulation

Recall: character value $\chi^\lambda(\mu)$ fulfills $s_\lambda = \sum_{\mu} \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu$.

$$\frac{n! \chi^\lambda(\pi)}{\dim \lambda} = \sum_{\sigma_2 \sigma_1 = \pi} (-1)^{\sigma_2} N'_{\sigma_1, \sigma_2}(\lambda),$$

where

**Definition**

$N'_{\sigma_1, \sigma_2}(\lambda)$ is the number of bijections $f : \{1, \ldots, n\} \cong \lambda$ such that for all $i$, $f(i)$ and $f(\sigma_1(i))$ (resp. $f(\sigma_2(i))$) are in the same row (resp. column).
Nice behaviour on short permutations

If \( \pi \in S_k \hookrightarrow S_n, (\pi(i) = i \ \forall \ i > k) \), then \( N'_{\sigma_1, \sigma_2}(\lambda) = 0 \) unless \( \sigma_1(i) = \sigma_2(i) = \pi(i) \ \forall \ i > k \).

In this case the formula becomes:

\[
\frac{n! \chi^\lambda(\iota(\pi))}{\dim \lambda} = \sum_{\substack{\sigma_1, \sigma_2 \in S_k \\
\sigma_2 \sigma_1 = \pi \ \text{and} \\
\sigma_1, \sigma_2 \in S_k}} (-1)^{\sigma_2} N'_{\iota(\sigma_1), \iota(\sigma_2)}(\lambda)
\]

But \( N'_{\iota(\sigma_1), \iota(\sigma_2)} = \# \{ f : \{1, \ldots, k\} \hookrightarrow \lambda \text{ with usual conditions} \} \cdot \frac{(n - k)!}{\text{choices of the places of } k+1, \ldots, n} \).
Nice behaviour on short permutations

Definition

\( N'_{\sigma_1,\sigma_2}(\lambda) \) is the number of injections \( f : \{1, \ldots, k\} \hookrightarrow \lambda \) such that, for all \( i \), \( f(i) \) and \( f(\sigma_1(i)) \) (resp. \( f(\sigma_2(i)) \)) are in the same row (resp. column).

\[
\Sigma_\pi(\lambda) := \frac{n \cdot (n - 1) \ldots (n - k + 1) \chi^\lambda(\iota(\pi))}{\text{dim } \lambda} = \sum_{\substack{\sigma_1,\sigma_2 \in S_k \\ \sigma_2 \sigma_1 = \pi}} (-1)^{\sigma_2} N'_{\sigma_1,\sigma_2}(\lambda).
\]
Forgetting injectivity

Definition

$N_{\sigma_1, \sigma_2}(T)$ is the number of functions $f : \{1, \ldots, k\} \rightarrow \lambda$ such that, for all $i$, $f(i)$ and $f(\sigma_1(i))$ (resp. $f(\sigma_2(i))$) are in the same row (resp. column).

$$\Sigma_\pi(\lambda) := \frac{n \cdot (n-1) \ldots (n-k+1) \chi^\lambda(\nu(\pi))}{\dim \lambda} = \sum_{\sigma_1, \sigma_2 \in S_k \atop \sigma_2 \sigma_1 = \pi} (-1)^{\sigma_2} N_{\sigma_1, \sigma_2}(\lambda).$$

Idea of proof: the total contribution of a non-injective function in rhs is easily seen to be 0.
Asymptotics is easy to read on this formula

Model: Fix a permutation \( \pi_0 \in S_k \) and a partition \( \lambda_0 \vdash k \)
Consider \( \pi = \iota(\pi_0) \) (i.e. we just add fixpoints) and \( \lambda = c \cdot \lambda_0 = \lambda \) multiplied by \( c \) (i.e. horizontal lengths are multiplied by \( c \))

Question: asymptotics of \( \frac{\chi_\lambda^\pi}{\dim \lambda} \) ?
Asymptotics is easy to read on this formula

Model: fix a permutation $\pi_0 \in S_k$ and a partition $\lambda_0 \vdash k$
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multiplied by $c$ (i.e. horizontal lengths are multiplied by $c$)

Question: asymptotics of $\frac{\chi_\lambda^\pi}{\dim \lambda}$?

Easy on the $N$’s: $N_{\sigma_1,\sigma_2}(c \cdot \lambda) = c^{\left| C(\sigma_2) \right|} N_{\sigma_1,\sigma_2}(\lambda)$
Asymptotics is easy to read on this formula

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Easy on the $N$’s: $N_{\sigma_1, \sigma_2}(c \cdot \lambda) = c^{|C(\sigma_2)|} N_{\sigma_1, \sigma_2}(\lambda)$
Dominant term of $\sum_{\pi} (\lambda)$:

$$N_{\pi, \text{Id}_k}(\lambda) = \prod_{\mu_i \in \text{type}(\pi)} \left( \sum_{j} \lambda_j^{\mu_i} \right)$$
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Question: asymptotics of $\frac{\chi_\lambda^\pi}{\dim \lambda}$?
Easy on the $N$'s: $N_{\sigma_1,\sigma_2}(c \cdot \lambda) = c|C(\sigma_1)| + |C(\sigma_2)| N_{\sigma_1,\sigma_2}(\lambda)$
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$\chi^\lambda(\pi) / \dim \lambda$?

Easy on the $N$'s: $N_{\sigma_1, \sigma_2}(c \cdot \lambda) = c|C(\sigma_1)| + |C(\sigma_2)| N_{\sigma_1, \sigma_2}(\lambda)$

Dominant term of $\sum_\pi(\lambda)$:

$$\sum_{\sigma_1 \sigma_2 = \pi} \pm N_{\sigma_1, \sigma_2}(\lambda)$$

$|C(\sigma_1)| + |C(\sigma_2)|$ maximal
Free cumulants

\[ \Sigma_\pi (c \cdot \lambda) = \sum_{\sigma_1 \sigma_2 = \pi, |C(\sigma_1)| + |C(\sigma_2)| \text{ maximal}} \pm N_{\sigma_1, \sigma_2}(\lambda) \]

But, \[ \left\{ \sigma_1 \sigma_2 = \pi, |C(\sigma_1)| + |C(\sigma_2)| \text{ maximal} \right\} \approx \prod NC_{\mu_i} \approx \prod \text{Trees}(\mu_i). \]

With generating series, one can prove (Rattan, 2006)

\[ \text{rhs} = \prod R_{\mu_i+1}(\lambda) \]

\( R_k \): free cumulants defined from the shape \( \omega_\lambda \) by Biane (1998).
Remarks

Works in more general context than sequences $c \cdot \lambda_0$ and $c \bullet \lambda_0$ (in fact, works as soon as a sequence of Young diagram has a limit)

These results were already known (Vershik & Kerov 81, Biane 98), but:

- we provide unified approach of both cases;
- our bound for error terms are better.
Description of the problem

Consider Plancherel’s probability measure on Young diagrams of size $n$

$$P(\lambda) = \frac{(\dim \lambda)^2}{n!}$$

Question: is there a limit shape for (renormalized rotated) Young diagram taken randomly with Plancherel’s measure when $n \to \infty$?
Normalized character values have simple expectations!

Fix $\pi \in \mathfrak{S}_n$. Let us consider the random variable:

$$X_\pi(\lambda) = \chi^\lambda(\pi) = \frac{\text{tr}(\rho_\lambda(\pi))}{\dim V_\lambda}.$$

Let us compute its expectation:

$$\mathbb{E}(X_\pi) = \frac{1}{n!} \sum_{\lambda \vdash n} (\dim V_\lambda) \cdot \text{tr}(\rho_\lambda(\pi))$$

$$= \frac{1}{n!} \text{tr} \left( \bigoplus_{\lambda \vdash n} V_\lambda^{\dim V_\lambda} \right)(\pi) = \frac{1}{n!} \text{tr}_{\mathbb{C}[\mathfrak{S}_n]}(\pi)$$

Last expression is easy to evaluate:

$$\mathbb{E}(X_\pi) = \delta_{\pi, \text{Id}_n}$$
Convergence of cumulants

Recall: we proved that $\prod_i R_{k_i+1} \approx \sum_{k_1, \ldots, k_r}$. Thus

$$\mathbb{E}(R_2) \approx n$$
$$\mathbb{E}(R_i) \approx 0 \text{ if } i > 2$$
$$\text{Var}(R_i) \approx 0 \text{ if } i \geq 2$$
Convergence of cumulants

Recall: we proved that \( \prod_i R_{k_i + 1} \approx \Sigma_{k_1, \ldots, k_r} \).

Thus

\[
\lim \mathbb{E} \left( \frac{R_2}{n} \right) = 1 \\
\lim \mathbb{E} \left( \frac{R_i}{\sqrt{n^i}} \right) = 0 \text{ if } i > 2 \\
\lim \text{Var} \left( \frac{R_i}{\sqrt{n^i}} \right) = 0 \text{ if } i \geq 2
\]

Easy to make it formal because \( R_k \in \text{Vect}(\Sigma_\pi) \).

\( \Rightarrow \) Random variables \( R_i / \sqrt{n^i} \) converge in probability towards the sequence \( (0, 1, 0, 0, \ldots) \).

General lemma from Kerov:

convergence of cumulants \( \Rightarrow \) convergence of Young diagrams
Existence of a limiting curve

Theorem (Logan and Shepp 77, Kerov and Vershik 77)

Let us take randomly (with Plancherel measure) a sequence of Young diagram $\lambda_n$ of size $n$. Then, in probability, for the uniform convergence topology on continuous functions, one has:

$$r_{45^\circ}(h_1/\sqrt{n}(\lambda_n)) \rightarrow \delta_\Omega,$$

where $\Omega$ is an explicit function drawn here:
Convergence of $q$-Plancherel measure

Case where expectation of character values are big:

- there can not be a limit shape after dilatation.
- we use the first approximation for characters $\sum_{\pi} (\lambda) \approx \prod \rho_{\mu_i}(\lambda)$. 
Convergence of $q$-Plancherel measure

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Example: $q$–Plancherel measure ($q < 1$)
- defined using representation of Hecke algebras
- one can prove

$$\mathbb{E}_{q}(\sum_{\pi}) = \frac{(1 - q)^{|\mu|}}{\prod_i 1 - q^{\mu_i}} n(n - 1) \ldots (n - |\mu| + 1)$$

Thus

$$\mathbb{E}_{q}(p_k) \approx \frac{(1 - q)^k}{\prod_i 1 - q^k} n^k \quad \text{Var}_{q}(p_k) \approx 0$$
Convergence of $q$-Plancherel measure

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- defined using representation of Hecke algebras
- one can prove

$$E_q(\sum_{\pi}) = \frac{(1 - q)^{|\mu|}}{\prod_i 1 - q^{\mu_i}} n(n - 1) \ldots (n - |\mu| + 1)$$

Thus

$$\lim E_q(p_k/n^k) = \frac{(1 - q)^k}{\prod_i 1 - q^k} \quad \lim \text{Var}_q(p_k/n^k) = 0$$
Theorem (F., Méliot, 2010)

Let $q < 1$. In probability, under $q$-Plancherel measure,

$$\forall k \geq 1, \quad \frac{p_k(\lambda)}{|\lambda|^k} \rightarrow_{M_{n,q}} \frac{(1 - q)^k}{1 - q^k}. $$

Moreover,

$$\forall i \geq 1, \quad \frac{\lambda_i}{n} \rightarrow_{M_{n,q}} (1 - q) q^{i-1};$$

We also obtained the second-order asymptotics.
End of the talk

Thank you for listening

Questions?