# Large permutations and permutons

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(joint work with F. Bassino, M. Bouvel,
L. Gerin, M. Maazoun and A. Pierrot)

CNRS, Université de Lorraine

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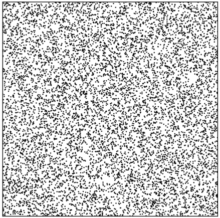
# Introduction

Main topic: random permutations

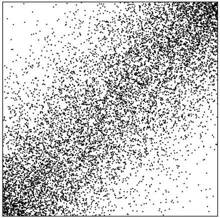
- Classical questions: look at some statistics, like the number of cycles (of given length), pattern occurrences, longest increasing subsequences, ...
  (usually for uniform, Ewens or Mallows distributions)
- a more recent approach: look for a limit for the rescaled permutation matrix; such limits are called permutons.
  (interesting for non-uniform models or constrained permutations)

This talk: presentation of the notion of permutons and of some convergence results for them.

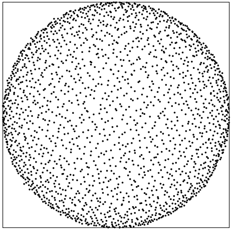
# A few random permutations



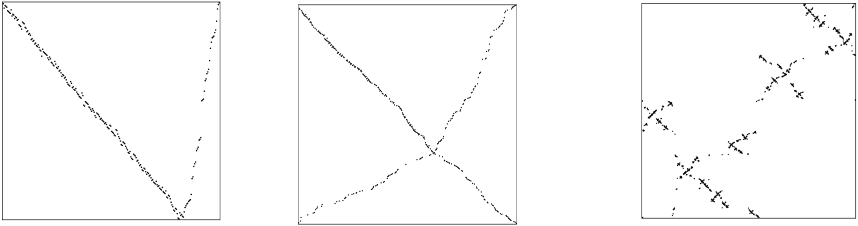
Uniform



Mallows ($\mathbb{P}(\sigma) \propto q^{\mathrm{inv}(\sigma)}$)



Sorting network,
half way (©AHRV '07)



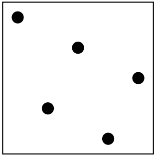
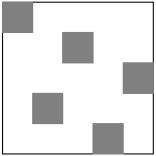
Uniform random pattern-avoiding permutations

# The theory of permutons

(starting from Hoppen, Kohayakawa, Moreira, Rath, Sampaio, '13)

# How to look at large permutations?

A permutation $\pi$ can be encoded as a probability measure $\mu_\pi$ on $[0,1]^2$.

$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

In $\mu_\pi$, each small square has weight $1/n$ (i.e. density $n$).

We have a natural notion of limit for such objects: the weak convergence.
This defines a compact Polish space.

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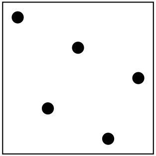
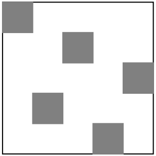
$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

In $\mu_\pi$, each small square has weight $1/n$ (i.e. density $n$).

Note: the projection on $\mu_\pi$ on each axis is the Lebesgue measure on $[0,1]$ (in other words, $\mu_\pi$ has uniform marginals).
$\rightarrow$ potential limits also have uniform marginals.

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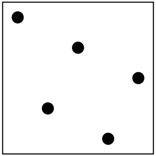
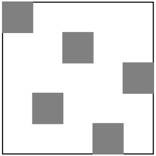
$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

In $\mu_\pi$, each small square has weight $1/n$ (i.e. density $n$).

**Definition**

A permuton is a probability measure on $[0,1]^2$ with uniform marginals.

Remark: permutons had been considered before by staticians under the name *copula*.

Next few slides: connection with permutation patterns.

# Permutation patterns

**Definition**

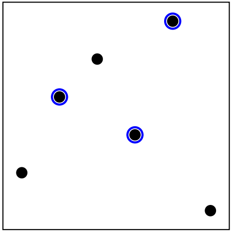
An occurrence of a pattern $\tau$ in $\sigma$ is a subsequence $\sigma_{i_1} \dots \sigma_{i_k}$ that is order-isomorphic to $\tau$, *i.e.* $\sigma_{i_s} < \sigma_{i_t} \Leftrightarrow \tau_s < \tau_t$.

Example (occurrences of 2 1 3)

2 4 5 3 6 1

8 2 3 4 6 1 7 5

Visual interpretation



# Pattern density in permutations and permutons

If $\tau$ and $\sigma$ are permutations of size $k$ and $n$, resp., we set

$$\widetilde{\mathrm{occ}}(\tau,\sigma) := \binom{n}{k}^{-1} \cdot \# \left\{ \begin{array}{c} \text{occurrences of} \\ \tau \text{ in } \sigma \end{array} \right\} \in [0,1].$$

In other terms: take $k$ elements uniformly at random in $\sigma$, the probability to find a pattern $\tau$ is $\widetilde{\mathrm{occ}}(\tau,\sigma)$.

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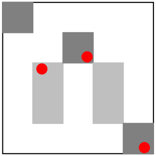
This probabilistic interpretation extends to permutons: replacing $\sigma$ with a permuton $\mu$

$$\widetilde{\mathrm{occ}}(\tau,\mu) := \mathbb{P}^{\mu}(U^{(1)}, \cdots, U^{(k)} \text{ form a pattern } \tau),$$

where $U^{(1)}, \cdots, U^{(k)}$ are i.i.d. points in $[0,1]^2$ with distribution $\mu$.



a "231 pattern" in a permuton

# Pattern density convergence and permuton convergence

Theorem (Hoppen, Kohayakawa, Moreira, Rath, Sampaio, '13)

*For each $n \geq 1$, let $\sigma_n$ be a permutation of size n. TFAE*

(a) $\mu_{\sigma_n}$ *converges to some permuton* $\mu$.

(b) *For every pattern* $\pi$, *the proportion* $\widetilde{\mathrm{occ}}(\pi, \sigma_n)$ *tends to some* $\delta_\pi$

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Theorem (Bassino-Bouvel-F.-Gerin-Maazoun-Pierrot, '17)

*For each $n \geq 1$, let $\boldsymbol{\sigma}_n$ be a random permutation of size n. TFAE*

(a) $\mu_{\boldsymbol{\sigma}_n}$ *converges in distribution to some random permuton* $\boldsymbol{\mu}$

(b) *For every pattern* $\pi$, *there is a* $\Delta_\pi \geq 0$ *such that*

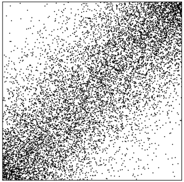
$$\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)] \xrightarrow{n\to\infty} \Delta_\pi.$$

Mallows model on $S_n$: $\mathbb{P}(\sigma_n) \propto q_n^{\mathrm{inv}(\sigma_n)}$,
where $\mathrm{inv}(\sigma) = \#\{(i,j) \text{ with } i < j \text{ and } \sigma(i) > \sigma(j)\}$.

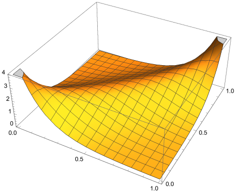
**Theorem (Starr, '09)**

*Take $q_n = 1 - \beta/n$. Then $\mu_{\boldsymbol{\sigma}^{(n)}}$ converge to the deterministic permuton with density*

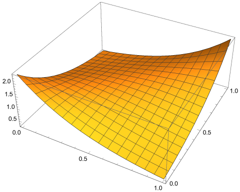
$$u(x,y) = \frac{(\beta/2)\sinh(\beta/2)}{\left(e^{\beta/4}\cosh(\beta[x-y]/2) - e^{-\beta/4}\cosh(\beta[x+y-1]/2)\right)^2}.$$



Simulation ($n = 10000$, $\beta = 6$)




$\beta = 6$




$\beta = 2$

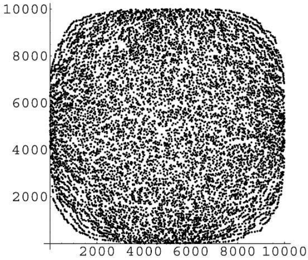
# Erdős-Szekeres extremal permutations

An Erdős-Szekeres extremal permutation is a permutation of size $n^2$ that has no monotone subsequence of size $n+1$.

Theorem (Romik, '06)

*Let $\boldsymbol{\sigma}_n$ be a uniform random Erdős-Szekeres extremal permutation of size $n^2$. Then $\boldsymbol{\sigma}_n$ converges to a deterministic permuton supported by*

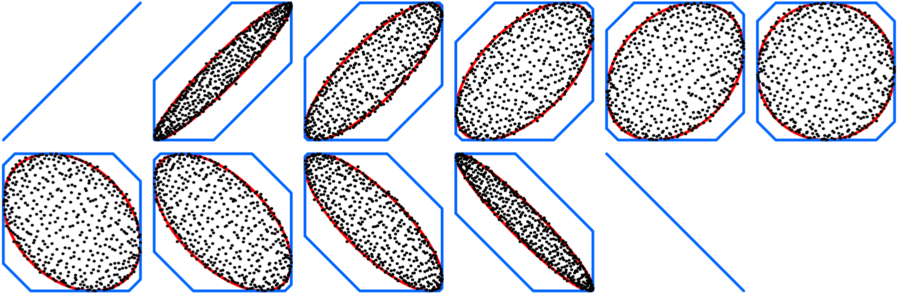
$$\left\{x, y \in [0,1]^2 : (x^2 - y^2)^2 + 2(x^2 + y^2) \leq 3\right\}$$




© Romik

# Random sorting networks

A sorting network is a minimal path going from the identity permutation to the reverse permutation, switching two adjacent entries at each step.



Random sorting network, ©Angel, Holroyd, Romik and Virag ('07)

A formula for the limiting process in the space of permutons was conjectured by Angel, Holroyd, Romik and Virag ('07) and proved by Dauvergne ('18).

# And more. . .

- Random permutations in grid classes (Bevan '15), Square permutations (Borga–Slivken '19), various exponentially biased models (Mukherjee '16, Bouvel–Nicaud–Pivoteau '19), . . .
- Large deviation principle for uniform random permutations in the space of permutons (Trashorras, '08, Kenyon–Král–Radin–Winkler, '15, Borga–Das–Mukherjee–Winkler, '22).
- Asymptotics of the number of cycles of fixed length (Mukherjee, '16), of the length of the longest increasing subsequence (Mueller–Starr, '13) and of the total displacement (Bevan–Winkler, '19) in Mallows permutations using the permuton limit.

# Limits of permutation classes with a finite specification

(joint work with Bouvel, Bassino, Gerin, Maazoun, Pierrot)

# Permutation classes

**Definition**

A set $\mathcal{C}$ of permutations (of all sizes) is a class if for all permutations $\pi$ in $\mathcal{C}$, and all *patterns* $\tau$ of $\pi$, $\tau$ is also in $\mathcal{C}$.

Equivalently, a class is the set of permutations avoiding given patterns.

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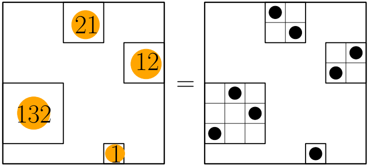
- Traditionally analyzed from an enumerative point of view: how many permutations of size $n$ are there in a given class?
- More recently from a probabilistic point of view: what does a uniform random permutation in a given class look like?
  (Atapour, Bevan, Borga, Dokos, Hoffman, Janson, Liu, Madras, Mansour, Miner, Pak, Pehlivan, Pinsky, Rizzolo, Slivken, Stufler, Yıldırım, . . . )

# Substitution in permutations

## Definition of substitution

Let $\theta$ be a permutation of size $d$ and $\pi^{(1)}, \dots, \pi^{(d)}$ be permutations. The diagram of the permutation $\theta[\pi^{(1)}, \dots, \pi^{(d)}]$ is obtained by replacing the $i$-th dot in the diagram of $\theta$ with the diagram of $\pi^{(i)}$ (for each $i$).

$2413[132, 21, 1, 12] =$




$= 24387156$

## Definition

A permutation is called simple if it cannot be obtained as a nontrivial substitution.

Examples: 12, 21, 3142, 2413, 25314, ...

# Classes with finitely many simple permutations

**Theorem (Albert, Atkinson, '05)**

*Every class $\mathcal{C}$ containing only finitely many simple permutations has a finite basis and an algebraic generating function.*

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*Every class $\mathcal{C}$ containing only finitely many simple permutations has a finite basis and an algebraic generating function.*

**Problem**

Let $\mathcal{C}$ be a class with finitely many simple permutations. Describe the permuton limits of a uniform random permutation in $\mathcal{C}$.

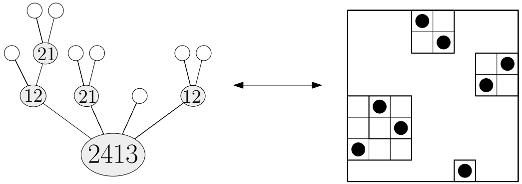
# Encoding permutations as trees

**Proposition (Albert, Atkinson, '05)**

*Every permutation $\sigma$ of size $n \geq 2$ can be uniquely decomposed as either:*

- $\alpha[\pi^{(1)}, \dots, \pi^{(d)}]$, *where $\alpha$ is simple of size $d \geq 4$,*
- $12[\pi^{(1)}, \pi^{(2)}]$, *where $\pi^{(1)}$ is* 12*-indecomposable,*
- $21[\pi^{(1)}, \pi^{(2)}]$, *where $\pi^{(1)}$ is* 21*-indecomposable.*

Iterating this with $\pi^{(1)}, \dots, \pi^{(d)}$, we can write any permutation as imbricated substitutions of simple permutations, which we represent as a tree (called substitution decomposition tree):

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# Classes with finitely many simple permutations (2/2)

**Theorem (Bassino-Bouvel-Pierrot-Pivoteau-Rossin '17; stated informally)**

*Fix a permutation class $\mathcal{C}$ with finitely many simple permutations. Then the set of trees corresponding to permutations in $\mathcal{C}$ can be defined via a finite system of equations.*

**Example: the system of equations defining $\mathcal{C} = \mathrm{Av}(132)$**

$$\begin{cases} \mathcal{C} & = \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\mathrm{not}\oplus}, \mathcal{C}_{\langle 21 \rangle}] \quad \uplus \quad \ominus[\mathcal{C}^{\mathrm{not}\ominus}, \mathcal{C}] \\ \mathcal{C}^{\mathrm{not}\oplus} & = \{\bullet\} \quad \uplus \quad \ominus[\mathcal{C}^{\mathrm{not}\ominus}, \mathcal{C}] \\ \mathcal{C}^{\mathrm{not}\ominus} & = \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\mathrm{not}\oplus}, \mathcal{C}_{\langle 21 \rangle}] \\ \mathcal{C}_{\langle 21 \rangle} & = \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\mathrm{not}\oplus}_{\langle 21 \rangle}, \mathcal{C}_{\langle 21 \rangle}] \\ \mathcal{C}^{\mathrm{not}\oplus}_{\langle 21 \rangle} & = \{\bullet\}. \end{cases}$$

$\oplus = 12$; $\ominus = 21$; not $\oplus$ means 12 indecomposable
$\langle 21 \rangle$ means "avoiding 21".

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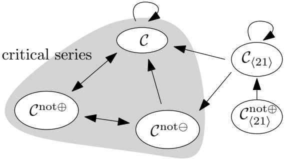



On the right: the dependency graph of the system with the classes of maximal growth rate in gray (called critical classes).

# Main theorem

**Theorem (BBFGMP, '19)**

*Let $\mathcal{C}$ be a family of permutations with a finite analytic specification (e.g. a permutation class with finitely many simple permutations). Assume that the dependency graph restricted to critical families is strongly connected (plus some weak aperiodicity assumption).*

# Main theorem

**Theorem (BBFGMP, '19)**

*Let $\mathcal{C}$ be a family of permutations with a finite analytic specification (e.g. a permutation class with finitely many simple permutations). Assume that the dependency graph restricted to critical families is strongly connected (plus some weak aperiodicity assumption).*

*essentially linear case* *If the specification contains no products of critical families, then a uniform random permutation in the class converges to an $X$-permuton with computable parameters.*

*essentially branching case* *If the specification contains a product of critical families, then a uniform random permutation in the class converges to a Brownian separable permuton with computable parameters.*

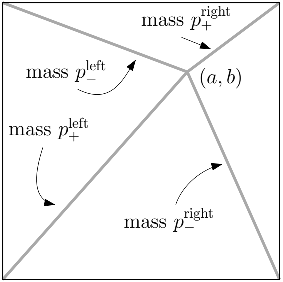
Description of the limit permutons and examples in the next few slides...

# The $X$-permuton

Parameter: a quadruple of sum 1

$$(p_+^{\text{left}}, p_+^{\text{right}}, p_-^{\text{left}}, p_-^{\text{right}}).$$

We set $a = p_+^{\text{left}} + p_-^{\text{left}}$
and $b = p_+^{\text{left}} + p_-^{\text{right}}$
(to ensure the uniform marginal condition).



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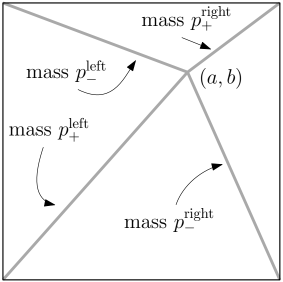
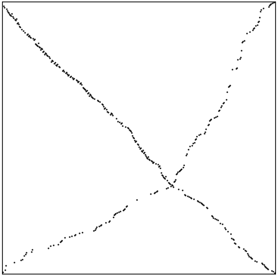
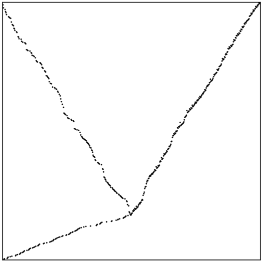



Note: this is a deterministic permuton.

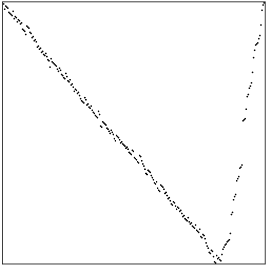
# The essentially linear case: examples



Av(2413, 3142,
2143, 34512)



Av(231, 21543)

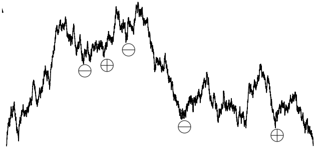


Av(2413, 1243,
2341, 41352, 531642)

Note: in the second (resp. third) case, one (resp. two consecutive) parameters are 0. Diagonals are also degenerate $X$-permutons (with 2 opposite or 3 parameters equal to 0).

# The Brownian separable permuton (construction by Maazoun '20)

Parameter: $p \in [0, 1]$




$(e, S)$

- $e$ is a Brownian excursion and $S : \text{LocalMin}(e) \to \{\oplus, \ominus\}$ is a independent assignment of signs to local minima of $e$ (the probability to get a $\oplus$ is $p$).

# The Brownian separable permuton (construction by Maazoun '20)

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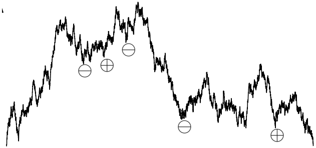
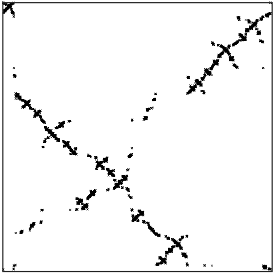



$(e, S)$

$\mapsto \quad \sigma \quad \mapsto$



$\boldsymbol{\mu} = (x, \sigma(x))_\star(\mathrm{Leb}([0,1])$

- $\sigma : [0,1] \to [0,1]$ is the unique Lebesgue preserving function s.t. $(x, y)$ is an inversion if and only if the sign of $\min_{[x,y]} e$ is $\ominus$.
- The Brownian separable permuton is the "graph of the function $\sigma$".

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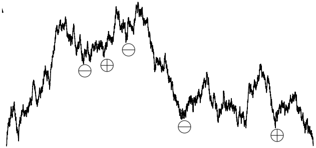
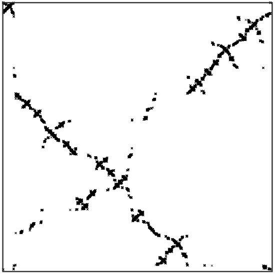


$(e, S)$

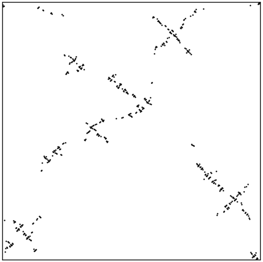
$\mapsto \quad \sigma \quad \mapsto$



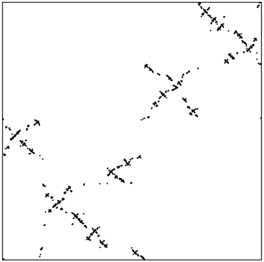
$\boldsymbol{\mu} = (x, \sigma(x))_\star(\mathrm{Leb}([0, 1]))$

Note: this a random permuton. No concentration phenomenon here.

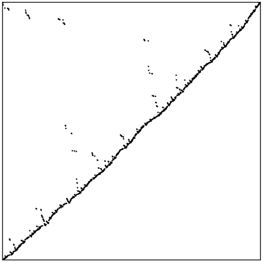
# The essentially branching case: examples



Av(2413, 3142)
separable permutations



Av(2413, 31452,
41253, 41352, 531246)



Av(231)

The limit in the last case is a degenerate Brownian permuton with $p = 1$, that is the diagonal of the square. This convergence to the diagonal (and much more precise results) was already known.

# A word on the proofs

1. Reminder: enough to prove that, for any $\tau$,

$$\mathbb{E}\big[\,\widetilde{\mathrm{occ}}(\tau,\boldsymbol{\sigma}_n)\big] \to \mathbb{E}\big[\,\widetilde{\mathrm{occ}}(\tau,\boldsymbol{\nu})\big],$$

where $\boldsymbol{\nu}$ is the targeted limit random permuton.

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2. The RHS can be evaluated easily (elementary for $X$-permuton, using some results on Brownian excursion for the Brownian one).

3. The LHS can be computed combinatorially:
$$\mathbb{E}[\widetilde{\text{occ}}(\tau,\boldsymbol{\sigma}_n)] = \frac{\#\{\sigma \in \mathcal{C}_n, I \subset [n] : \text{pat}_I(\sigma) = \tau\}}{\binom{n}{k}\;|\mathcal{C}_n|}.$$
We will estimate that through analytic combinatorics.

# Analytic combinatorics

The strongly connectedness hypothesis ensures that

- in the essentially linear case,

$$C(z) \sim a \frac{1}{1-\frac{z}{\rho}}, \text{ implying } |\mathcal{C}_n| \sim a \rho^{-n}.$$

- in the branching case,

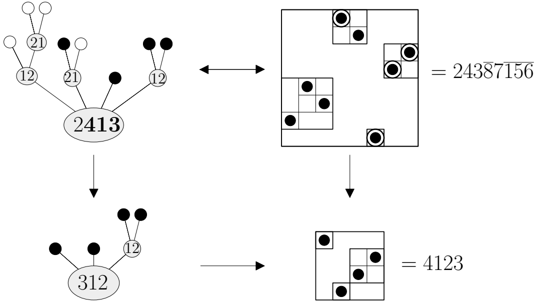
$$C(z) \sim a - b\sqrt{1-\tfrac{z}{\rho}}, \text{ implying } |\mathcal{C}_n| \sim \frac{b}{2\sqrt{\pi}} n^{-3/2} \rho^{-n}$$

The difficulty is to estimate

$$\{\#\{\sigma \in \mathcal{C}_n, I \subset [n] : \mathrm{pat}_I(\sigma) = \pi\}\}.$$

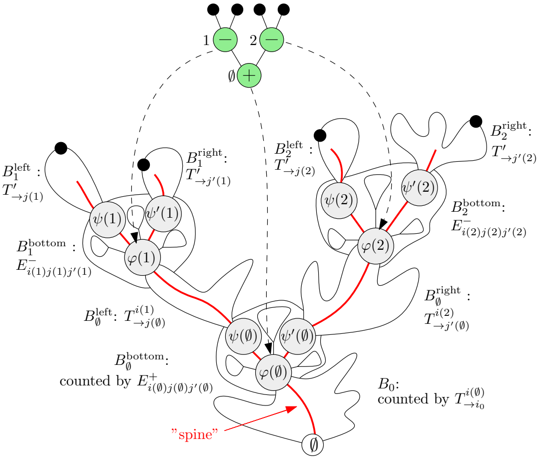
# From permutations to trees

Patterns in permutations correspond to “induced subtrees” in their decomposition tree :




Instead of counting permutations with an occurrence of a given pattern, we count tree with marked leaves inducing a given (decorated) subtree.

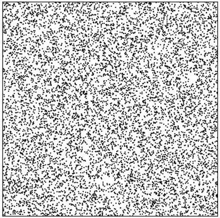
# A picture of a combinatorial decomposition



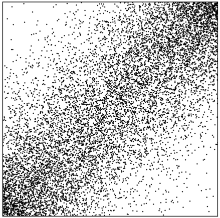

.

This translate into a formula for the series of a tree with marked leaves inducing a given subtree and we can study its behaviour at the singularity...

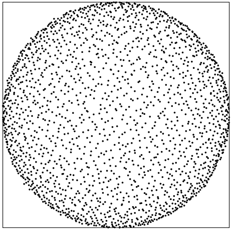
# Thank you for your attention



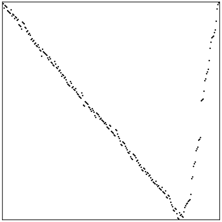
Uniform

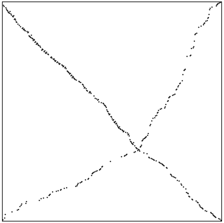


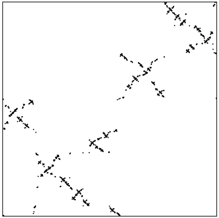
Mallows ($\mathbb{P}(\sigma) \propto q^{\mathrm{inv}(\sigma)}$)



Sorting network,
half way (©AHRV '07)





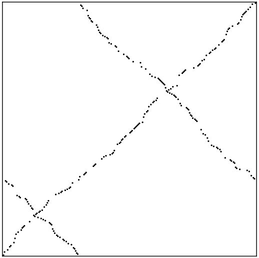


Uniform random pattern-avoiding permutations

# Extra slide 1: is the strong connectivity condition necessary?

Yes!

Here is a class with no simple permutations and a “double X” limit:



Av(2413, 3142, 3412, 214365, 52143, 32541)

We can treat such examples on a case-by-case basis from their finite specification, but we have no general theorem!

# Extra slide 2: the *intensity* of the Brownian permuton

Since the Brownian permuton $\boldsymbol{\mu}_p$ is a random measure, we can consider its intensity measure $\mathbb{E}\boldsymbol{\mu}_p$, defined by

$$(\mathbb{E}\boldsymbol{\mu}_p)(R) = \mathbb{E}(\boldsymbol{\mu}(R)), \text{ for any rectangle } R \subseteq [0,1]^2.$$

Theorem (Maazoun '20)

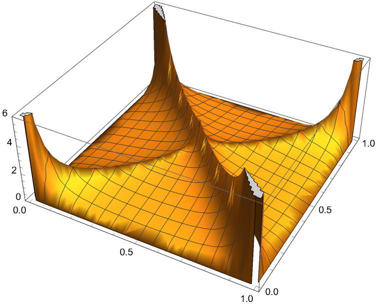
*The intensity measure $\mathbb{E}\boldsymbol{\mu}_p$ has density w.r.t to Lebesgue measure*

$$f_p(x,y) = \int_{\max(0,x+y-1)}^{\min(x,y)} \frac{3p^2(1-p)^2 da}{2\pi(a(x-a)(1-x-y+a)(y-a))^{3/2}\left(\frac{p^2}{a} + \frac{(1-p)^2}{(x-a)} + \frac{p^2}{(1-x-y+a)} + \frac{(1-p)^2}{(y-a)}\right)^{5/2}}.$$

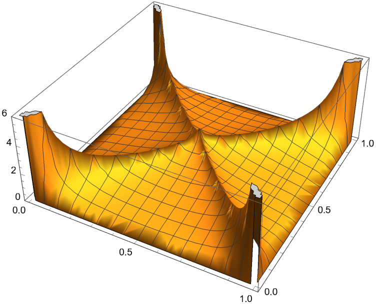
Concretely, if $\boldsymbol{\sigma}_n$ tends to $\boldsymbol{\mu}_p$, then, for any rectangle $R \subseteq [0,1]^2$

$$\mathbb{E}\big[\#\{(i,j) \in nR : \sigma(i) = j\}\big] \sim n \int_{(x,y)\in R} f_p(x,y)dxdy.$$

# Extra slide 2bis: picture of $\mathbb{E}\boldsymbol{\mu}_p$



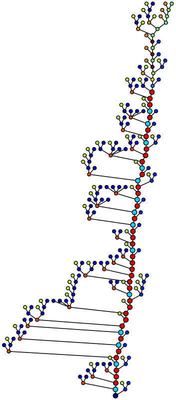

density of $\mathbb{E}\boldsymbol{\mu}_{.4}$




density of $\mathbb{E}\boldsymbol{\mu}_{.5}$

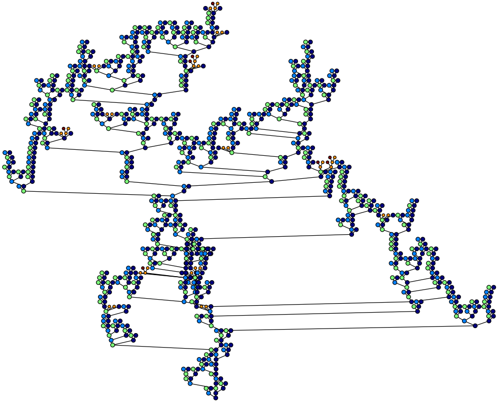
For $p = .5$, this function was found (under a different form) by Pak and Dokos, in the context of doubly alternating Baxter permutations.

# Extra slide 3: underlying random trees



essentially linear case

Av(2413, 1243, 2341, 41352, 531642)



essential branching case

Av(2413, 31452, 41253, 41352, 531246)