Goal: characterize polynomials in infinitely many variables s.t.
\[ f(x_1, x_2, \ldots) \big|_{x_i = x_{i+1}} = f(x_1, \ldots, x_{i-1}, x_{i+2}, \ldots) \]  

The main problem

Motivation: smooth functions on Young diagrams

- A virtual alphabet

We define
\[ \mathrm{WQSym} \]
and a virtual alphabet

Let \( M_1 \) be the monomial basis of \( \mathrm{QSym} \), \( S_i \) set of generators of \( \mathrm{Sym} \).

Notation: \( M_1 \) monomial basis of \( \mathrm{QSym} \); \( S_i \) set of generators of \( \mathrm{Sym} \).

\[ \sigma_i(x)A = \sum_i M_i(x)S_i(A) = \prod_{i \geq 1} \sigma_i(A)^{(-1)^i}, \]

where \( \sigma_i(A) = 1 + x_i S_1 + x_i^2 S_2 + x_i^3 S_3 + \ldots \)

For example,
\[ M_i(\lambda) = -x_i^4 + x_i^3 + x_i^2 - x_i - \ldots \]

The \( M_1(\lambda) \) indeed remain the same when one puts \( x_{k+1} = x_k \).

First result: solution to (1) in the commutative framework

A function \( f \) satisfies the functional equation (1) if and only if \( f \in \mathrm{QSym}(\mathbb{X}) \).

Links with other results

- Natural generalization of the algebra of polynomial functions on Young diagrams considered by Kerov and Olshanski (which corresponds to \( \mathrm{Sym}(\mathbb{X}) \)).
- Also extends a result of Stembridge.
- Stembridge’s problem: find solutions of (1) which are in addition symmetric in the odd-indexed variables and separately in the even-indexed variables.
- Stembridge’s solution: symmetric functions evaluated on \( \mathbb{X} \).

Noncommutative generalization: Solve (1) when \( f \) (written \( P \)) is now a noncommutative polynomial

\( \mathrm{WQSym} \) and a virtual alphabet

Notation: \( P_\alpha \) monomial basis of \( \mathrm{WQSym} \), indexed by packed words.

We define \( P_\alpha(\lambda) \) as:

- If \( \alpha \) is nondecreasing, \( P_\alpha(\lambda) \) is the noncommutative analogue of \( M_{\mathrm{val}(\alpha)}(\lambda) \), where \( \alpha_k \) is replaced by \( \alpha_k \) and all letters in any monomial of \( P_\alpha(\lambda) \) are in nondecreasing order.

- For other \( \alpha \), define \( P_\alpha \) by an action of the symmetric group.

\[ P_{112}(\lambda) = \sum_i a_{2i+1}^3 + \sum_{i < j} (-1)^{i+j} a_i a_j a_k; \]

- For other \( \alpha \), define \( P_\alpha \) by an action of the symmetric group.

\[ P_{121}(\lambda) = \sum_i a_{2i+1}^3 + \sum_{i < j} (-1)^{i+j} a_i a_j a_k; \]

is obtained by swapping the second and third letter in each monomial of \( P_{112}(\lambda) \).

Third result: solution to (1) in the commutative framework

A noncommutative polynomial \( P \) satisfies (1) if and only if \( P \in \mathrm{WQSym}(\lambda) \).

On virtual alphabets for \( \mathrm{WQSym} \)

- There is no formula analogous to (2) to define \( P_\alpha(\lambda) \).
- It is therefore surprising that such a simple definition of \( P_\alpha(\lambda) \) works.
- The functional equation (1) helps proving that \( F \rightarrow F(\lambda) \) defines an algebra morphism!

Fourth result: a combinatorial by-product

Let \( \mathbb{K} \) be the smallest two-sided ideal of \( \mathrm{WQSym} \) containing \( P_1 \) and whose homogeneous components \( \mathbb{K}_n \) are stable by the action of \( S_n \). Then the dimension of \( \mathbb{K}_n \) is the number of set-compositions of \( \{1, \ldots, n\} \) with an odd number of parts.