Shifted Schur functions

Sahi, Biedenharn-Louck, Okounkov, Olshanski, 90's
a result joint with P. Alexanderson at the end

Context: Schur polynomials $s_\lambda(x_1, \ldots, x_N)$ ($x+\lambda$)

- defined as a quotient of determinants
- as sum over $SSYT$

Other approach developed in the 80's:

consider interpolation symmetric polynomial

$\rightarrow$ see again Schur functions appear...

$\rightarrow$ we will conclude by a new positivity result on these interpolation sym pl.

I Interpolation symmetric polynomial.

Idea: define sym functions by values at specific points

(like interp. pol.)

How many points? to get sym pol. deg $\leq d$,

a priori $|P^\leq d|$ points

A not any set of points will work. part. size at most $d$ length $- N$

natural choices: $P^\leq d$

We will add shifts: fix parameters $e_1, \ldots, e_N$

let $M_e := \{ x^e : x^e = (e_1x_1, \ldots, e_Nx_N) ; \ \| e \| \leq d \}$
Then let \( d \geq 1 \) and \( e \in \mathbb{C}^n \).

Assume \( 1 \leq i < j \), \( e_i - e_j \neq -1, -2, ..., -\left\lfloor \frac{d}{i-j} \right\rfloor \).

Then for every map \( f : M_e \to \mathbb{C} \), there exists a unique symmetric polynomial \( f \) of degree \( \leq d \) such that \( f/M_e = f \).

Proof: suppose \( f = \sum_{\lambda \leq d} a_{\lambda} m_{\lambda} \).

Condition \( f/M_e = f \) is a square system of equations \( a_{\lambda} \).

Existence for all \( f \) implies uniqueness. (\( = \) det square matrix \( \neq 0 \))

Proof of \( \alpha \) by induction on \( n \) and \( d \).

Look for \( f \) as:

\[
f(x) = \prod_{i=1}^{n} (x_i - e_i) \cdot R(x) + P(x) \quad \text{sym polynomial}
\]

when \( x_n = e_n \) (i.e. \( x_n = 0 \)), first term vanishes so we want \( g/M_{e_1 \ldots e_{n-1} 0^d} = f \).

Induction \( n \leq d \), \( (e_1, \ldots, e_{n-1}) \).

\[
g(x) = (x_{n-1} - e_{n-1}) \ldots (x_1 - e_1) \quad \text{sym polynomial}
\]

(we cheated: IH says there exists \( g \) sym in \( n-1 \) variables and we want \( g \) sym in \( n \) variables)

but simple to create a sym pol. in \( n \) var from sym pol. in \( n-1 \) variables)

\[
\text{when } x_n \neq e_n \text{ rewrite } (x_n) \text{ as } \frac{f - P(x_n)}{M_{e_1 \ldots e_{n-1}}}
\]

\[
\text{we want } f/M_{e_1 \ldots e_{n-1} 0^d} = f
\]
\[ g \text{ has been chosen; we ensured that } g/\pi_{\Omega < \mathbb{N}^+} = f \]

\[ \implies \text{we still need to ensure } g/\pi_{\mathbb{N}^+ \cap \Omega > \mathbb{N}^+} = f \text{ on } \mathbb{N}^+ \cap \Omega > \mathbb{N}^+ \]

\[ \text{i.e. } g/\pi_{\mathbb{N}^+ \cap \Omega > \mathbb{N}^+} = \frac{g - g}{\pi(x_i - x_j)} \]

\[ \text{Induction} \Rightarrow \text{there exists } \gamma, \eta \text{ such a sym poly. } \]

\[ \text{conclude existence of } f \text{ and the whole proof } \]

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**Particular case:** \( e_i = \mathbb{N}^+ \). Fix \( \mu \) of size \( d \).

There exists, up to a multiplicative constant, a unique \( t_\mu \) s.t.

\[ t_\mu(x_i) = 0 \quad \text{if} \quad |\mu| = |\lambda| \quad x \neq \mu \]

\[ t_\mu(\mu) \neq 0. \]

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**II Two formulas for \( t_\mu \)**

\[ \text{A. Determinantal formula} \]

Prop: \( t_\mu(x_1, \ldots, x_N) = \frac{\det((x_i)_k^\mu + N - j)}{\prod (x_i)_k^\mu} \)

Proof: \( \gamma \delta \text{ sym as quotient of two anti-sym. polynomials} \)

Vanishing of RNS for \( \lambda_i = \lambda + N - i \quad |\lambda| = |\mu| \quad |\lambda| \neq |\mu| \)

Assume \( \lambda_{i_0} < \mu_{i_0} \quad (x_i)_k^\mu + N - j = 0 \quad \text{if} \quad j < i_0 \)

Thus \( (x_i)_k^\mu + N - j = 0 \) \Rightarrow \text{det is } 0. \]
Cor: \( t_\mu = s_\mu + \text{Smaller degree term} \)

Cor: \( t_\mu(2) = 0 \) unless \( \lambda \supseteq \mu \).
(extra-vanishing property).

B. Combinatorial formula

Def: A reverse semi-standard Young tableau (RSST) of shape \( \lambda \) is

\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 3 \\
\end{array}
\]

\( \text{RSST}(\lambda, N) = \text{set of RSST of shape } \lambda \) with entries at most \( N \)

If \( \lambda \in \mathcal{A} \), then \( c(\lambda) = \text{column index - row index} \)

Prop: \( t_\mu(x_1, \ldots, x_N) = \sum \prod_{T \in \text{RSST}(\lambda,N)} \left( \frac{1}{\text{det}(T)} \text{det}(\text{minor}(-c(\lambda))) \right) \)

Step 1: RHS sym (exercise)

Hint: show that it's symmetric in \( x_i \) and \( x_{i+1} \)

Focus on boxes containing \( i \) and \( i+1 \)

Possibly easy to check its symmetric

Easy to check scan is symmetric
Vanishing property

\[ t_\mu(\lambda + \delta) = \sum_{\text{TERS}(\mu)} \frac{\prod_{\lambda_T} (\lambda_T - \lambda(\mu))}{\det_\mu \text{RSYT}(\mu)} \]

\[ \prod_{\det} (\lambda_T - \lambda(\mu)) \neq 0 \]

\[ \Rightarrow \lambda_T(i) \geq i \quad \text{but} \quad T(i, i) > \mu(i) \]

(for all \( i \))

\[ \Rightarrow \lambda_T(i) - i \geq \mu(i) \]

\[ \Rightarrow \lambda_T \geq \mu. \]

### A new positivity property

Prop (see \( \lambda(\mu) \geq 0 \)).

Thm (F., Alexandersson 2015):

\[ t_\mu(\lambda + \delta) = \sum \text{has nonnegative coefficients in the basis} \]

\[ \left( (\lambda_1 - \lambda_2)a_1, \ldots, (\lambda_{N-1} - \lambda_N)a_{N-1}, (\lambda_N)a_N \right) \]

lift to polynomials the positivity property above \( a_1, \ldots, a_N \geq 0 \)

\( \Rightarrow \) \( x \)-deformation shifts \( \ell_i = \frac{1}{x} \)

No explicit formula. But \( t_{\mu}(\lambda) = f_{\mu}(\lambda) + \text{smaller degree terms} \)

We conjecture a similar nonnegativity property.