Tree-indexed sums of Catalan numbers

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joint work with Paul Thévenin and Alin Bostan

CNRS, Institut Élie Cartan de Lorraine (IECL)

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Problem statement

For a tree T, we consider

$$S(T) := \sum_{(x_e)\in \mathbb{Z}_+^{E(T)}} \left(\prod_{v\in V(T)} \mathrm{Cat}_{X_v}\, t^{X_v} \right),$$

where $\mathrm{Cat}_k = \frac{1}{k+1}\binom{2k}{k}$ and $X_v = \sum_{e \ni v} x_e$.

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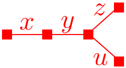
Example

Let $T =$ 

, then

$$S(T) = \sum_{x,y,z,u \geq 0} \mathrm{Cat}_x\, \mathrm{Cat}_{x+y}\, \mathrm{Cat}_{y+z+u}\, \mathrm{Cat}_z\, \mathrm{Cat}_u\, t^{2x+2y+2z+2u}.$$

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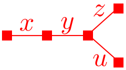
Let $T =$ 

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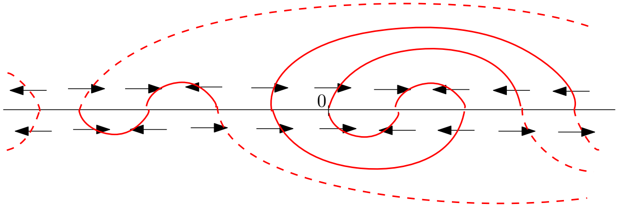
Our goal: compute these sums, and in particular the value at $t = 1/4$.

Motivation: meandric systems (1/5)

Definition (The Uniform Infinite Meandric System, or Infinite Noodle Soup)

Draw two bi-infinite sequences of i.i.d. left/right arrows and close them in the unique non-crossing way.



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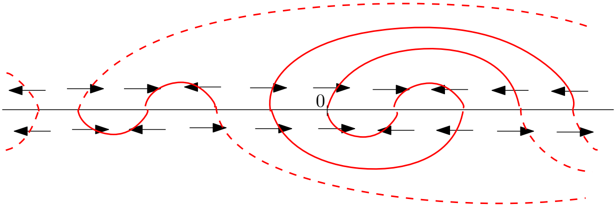

Questions

Is there an infinite connected component? What is the tail behaviour of the size of the component of 0? Can we at least compute $\mathbb{P}(|C_0| = k)$ for small k?

Motivation: meandric systems (2/5)

Conjectures

- Almost surely, there is no infinite component.
  (Curien–Kozma–Sidoravicius–Tournier, '19)
- As k tends to $+\infty$, we have $\mathbb{P}(|C_0| \geq k) \sim k^{-(2\sqrt{2}-1)/7+o(1)}$.
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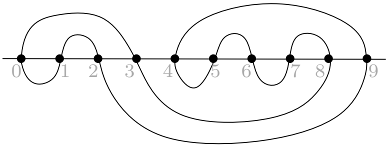

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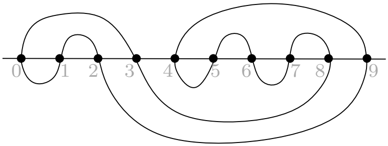


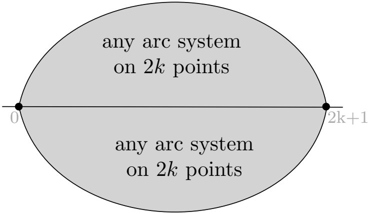
Conjecture (Di Francesco–Golinelli–Guitter, '00):

$$\#\{\text{meanders of size } 2n\} \sim C\, A^n n^{-\alpha},$$

with $\alpha = \dfrac{29 + \sqrt{145}}{12}$.

Motivation: meandric systems (3/5)

A configuration with $|C_0| = 2$ looks like



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$$\mathbb{P}[|C_0| = 2] = 2 \sum_{k \geq 1} \mathrm{Cat}_k^2 \, 2^{-4k-4}$$

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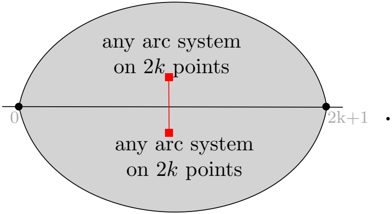

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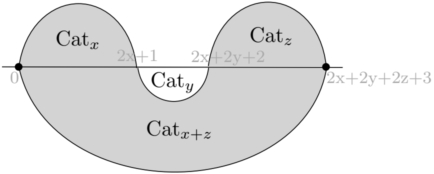
$$\mathbb{P}[|C_0| = 2] = 2 \sum_{k \geq 1} \mathrm{Cat}_k^2 \, 2^{-4k-4} = \tfrac{1}{8} \, S\left(\blacksquare\!\!-\!\!-\!\!\blacksquare \right).$$

Motivation: meandric systems (4/5)

Up to symmetry, a configuration with $|C_0| = 4$ looks like



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$$\mathbb{P}[|C_0| = 4] = 4 \times 2 \times \sum_{x,y,z \geq 0} \mathrm{Cat}_x \, \mathrm{Cat}_y \, \mathrm{Cat}_z \, \mathrm{Cat}_{x+z} \, 2^{-4x-2y-4z-8}$$

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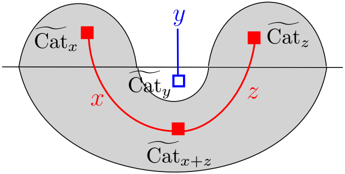

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$= \frac{1}{32} S\big($  $\big) \, S\big($  $\big)$.

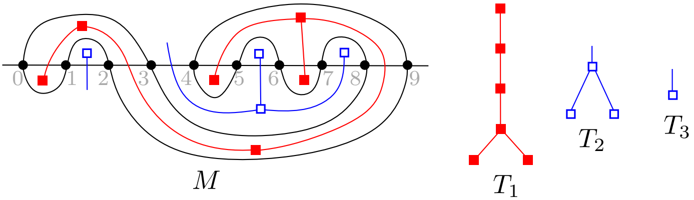
Motivation: meandric systems (5/5)

More generally, to compute $\mathbb{P}[|C_0| = k]$,

- we sum over all possible component "shapes", i.e. over meanders of size k;
- for a meander M,

$$\mathbb{P}(C_0 \simeq M) = 2^{-4k+1} k \prod_{i=1}^{d} S(T_i),$$

where the T_i's are the "dual trees" of the meander.



What formal calculus can do...

1. Compute specific values (P_n and S_n are respectively the path and the star with n vertices):

$$S(P_2)(\tfrac{1}{4}) = \frac{16}{\pi} - 4; \qquad S(P_3)(\tfrac{1}{4}) = 8 - \frac{64}{3\pi}; \qquad S(S_4)(\tfrac{1}{4}) = \frac{64}{15\pi};$$

$$S(P_4)(\tfrac{1}{4}) = -32 + \frac{64}{\pi} + \frac{128}{\pi^2}; \qquad S(P_5)(\tfrac{1}{4}) = ??$$

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2. A general structure theorem

Proposition (Corollary of Bostan–Lairez–Salvy, '17)

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Moreover *creative telescoping* yields an algorithm way to compute a (minimal) annihilating operator for $S(T)$;
but there is no algorithm to compute an explicit expression for $S(T)$.

Hypergeometric function and Gauss identity

Recall the definition of hypergeometric functions

$$ {}_2F_1(a,b;c;z) := \sum_{n\geq 0} \frac{a^{\uparrow n} b^{\uparrow n}}{c^{\uparrow n}} \frac{z^n}{n!}, $$

with $u^{\uparrow n} := u(u+1)\cdots(u+n-1)$.

Lemma (Gauss identity)

If $c-a-b>0$, we have

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

Our main result

Theorem

Let T be a tree, possibly with a half-edge. Then $S(T)$ belongs to the space

$$\mathbb{Q}\Big[t^{\pm 1}, \sqrt{1-4t}, {}_2F_1\big(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; 16t^2\big), {}_2F_1\big(-\tfrac{1}{2}, \tfrac{1}{2}; 2; 16t^2\big)\Big].$$

Moreover, ${}_2F_1\big(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; 16t^2\big)$ *and* ${}_2F_1\big(-\tfrac{1}{2}, \tfrac{1}{2}; 2; 16t^2\big)$ *are algebraically independent over* $\mathbb{Q}[t^{\pm 1}, \sqrt{1-4t}]$.

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- With this structure theorem, we can solve the annihilating operator and find an explicit expression for $S(T)$ → data in the paper.
- Our proof is constructive and yields another algorithm to compute $S(T)$ (not implemented).

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Corollary

For any tree T possibly with a half-edge, $S(T)(\frac{1}{4})$ belongs to $\mathbb{Q}[\frac{1}{\pi}]$.
Consequence on the uniform infinite meandric system: for any k, $\mathbb{P}(|C_0| = k)$ is a polynomial in $1/\pi$.

$S(P_2)$ and hypergeometric functions

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Reminder: one has the recurrence $(x+2)\,\mathrm{Cat}_{x+1} = 2(2x+1)\,\mathrm{Cat}_x$. Hence the quotient u_{x+1}/u_x is a rational function in x. Such terms are called hypergeometric. Typical hypergeometric sums are

$$ {}_2F_1(a,b;c;z) = \sum_{n \geq 0} \frac{a^{\uparrow n} b^{\uparrow n}}{c^{\uparrow n}} \frac{z^n}{n!}, $$

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Gauss identity yields ${}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1\right) = \frac{4}{\pi}$ and $S(P_2)(\frac{1}{4}) = \frac{16}{\pi} - 4$.

$S(P_3)$ and the quadratic recurrence

We want to compute $S(P_3) = \sum_{x,y \in \mathbb{Z}_+} \mathrm{Cat}_x \, \mathrm{Cat}_y \, \mathrm{Cat}_{x+y} \, t^{2x+2y}$.

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Rewrite the sum using $Z = x + y$.

$$\begin{aligned} S(P_3) &= \sum_{Z \geq 0} \mathrm{Cat}_Z \, t^{2Z} \left(\sum_{\substack{x,y \geq 0 \\ x+y=Z}} \mathrm{Cat}_x \, \mathrm{Cat}_y \right) \\ &= \sum_{Z \geq 0} \mathrm{Cat}_Z \, \mathrm{Cat}_{Z+1} \, t^{2Z}. \end{aligned}$$

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Again, this can be related to hypergeometric functions, namely

$$S(P_3) = \frac{1}{2t^2} \left(1 - {}_2F_1\left(-\tfrac{1}{2}, \tfrac{1}{2}; 2; 16t^2\right)\right),$$

and Gauss identity allows to compute

$$S(P_3)(\tfrac{1}{4}) = 8 - \frac{64}{3\pi}.$$

$S(P_2^h)$ and manipulating inequalities

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By symmetry, we also have

$$S(P_2^h) = \sum_{x \geq z \geq 0} \mathrm{Cat}_x \, \mathrm{Cat}_z \, t^{x+z}.$$

and thus

$$2S(P_2^h) = \sum_{x,z \geq 0} \mathrm{Cat}_x \, \mathrm{Cat}_z \, t^{x+z} + \sum_{\substack{x,z \geq 0 \\ x=z}} \mathrm{Cat}_x \, \mathrm{Cat}_z \, t^{x+z} = \Big(\sum_{x \geq 0} \mathrm{Cat}_x \, t^x \Big)^2 + S(P_2)$$

$$= \frac{1}{4t^2} \left(1 - \sqrt{1-4t}\right)^2 + \frac{1}{4t^2} \left(-1 + {}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; 16t^2\right)\right).$$

From edge-variables to vertex variables

We root T and write $v' \leq_T v$ if v' is a descendent of v. Color white (resp. black) vertices at even (resp. odd) distance from the root.

Lemma

The map sending $(x_e)_{e \in E(T)}$ to $(X_v)_{v \in V(T)}$ where $X_v = \sum_{e \ni v} x_e$, is injective and its range is the set of vectors $(X_v)_{v \in V(T)}$ satisfying

$$(*) \begin{cases} \sum_{w \in V_\circ(T)} X_w = \sum_{b \in V_\bullet(T)} X_b \\ \text{for all } v \in V_\circ(T),\ \sum_{w \in V_\circ(T), w \leq_T v} X_w \geq \sum_{b \in V_\bullet(T), b \leq_T v} X_b \\ \text{for all } v \in V_\bullet(T),\ \sum_{w \in V_\circ(T), w \leq_T v} X_w \leq \sum_{b \in V_\bullet(T), b \leq_T v} X_b. \end{cases}$$

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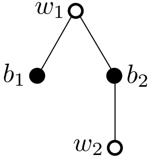
Corollary

$$S(T) = \sum_{\substack{(X_v) \in \mathbb{Z}_+^{V(T)} \\ (*)}} \prod_{v \in V} \mathrm{Cat}_{X_v}\, t^V.$$

Next example

A rooted version of P_4 is 

. We get

$$S(P_4) = \sum_{\substack{w_1, w_2, b_1, b_2 \geq 0 \\ w_1 + w_2 = b_1 + b_2, \;\; b_2 \geq w_2}} \mathrm{Cat}_{w_1} \mathrm{Cat}_{w_2} \mathrm{Cat}_{b_1} \mathrm{Cat}_{b_2} \, t^{w_1 + w_2 + b_1 + b_2}.$$

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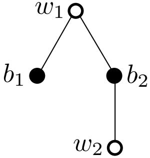
A rooted version of P_4 is 

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Manipulating the inequalities,

$$2S(P_4) = \sum_{\substack{w_1,w_2,b_1,b_2 \geq 0 \\ w_1+w_2=b_1+b_2}} \mathrm{Cat}_{w_1}\,\mathrm{Cat}_{w_2}\,\mathrm{Cat}_{b_1}\,\mathrm{Cat}_{b_2}\,t^{w_1+w_2+b_1+b_2}$$
$$+ \sum_{\substack{w_1,w_2,b_1,b_2 \geq 0 \\ w_1+w_2=b_1+b_2, b_2=w_2}} \mathrm{Cat}_{w_1}\,\mathrm{Cat}_{w_2}\,\mathrm{Cat}_{b_1}\,\mathrm{Cat}_{b_2}\,t^{w_1+w_2+b_1+b_2}.$$

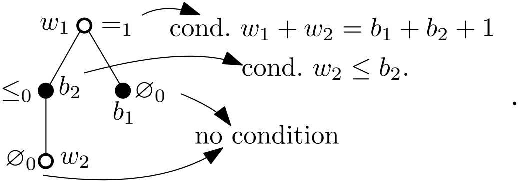
The first term can be computed using the quadratic recurrence, and the second is $S(P_2)^2$.

The general framework

- We consider rooted trees $\widetilde{T}$ with black/white/gray* vertices and vertex decorations in $\{=, \leq, \geq, \varnothing\} \times \mathbb{Z}$.
- To each vertex is associated a condition (C_v) comparing the sums of white variables and black variables below it.
- We consider the sum

$$S(\widetilde{T}) := \sum_{(X_v) \in \mathbb{Z}_+^{V_\bullet(T) \cup V_\circ(T)}} \left(\prod_{v \in V_\bullet(T) \cup V_\circ(T)} \mathrm{Cat}_{X_v}\, t^{X_v} \right) \left(\prod_v \mathbf{1}[C_v] \right).$$

Example: A decorated version of P_4



* Gray vertices do not have associated variables but may impose conditions.

Our main result (general version)

Theorem (Bostan, F., Thévenin, '25)

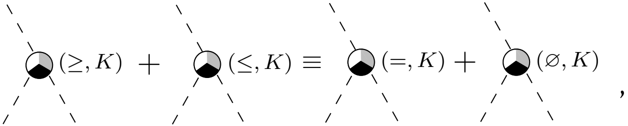
For any decorated tree $\widetilde{T}$, the series $S(\widetilde{T})(t)$ belongs to the space

$$\mathbb{Q}\Big[t^{\pm 1}, \sqrt{1-4t}, {}_2F_1\big(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; 16t^2\big), {}_2F_1\big(-\tfrac{1}{2}, \tfrac{1}{2}; 2; 16t^2\big)\Big].$$

Moreover, if the root decoration of $\widetilde{T}$ is $(=, K)$ for some K, then we can remove the generator $\sqrt{1-4t}$ above.

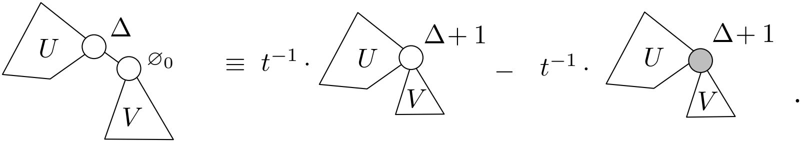
Some words on the proof

We have some reduction rules, like



(manipulating inequalities)

or



(using the quadratic recurrence)

Reducing any tree with such rules?

Let T be a bicolored tree. We start from the leaves.

- If there are two twin leaves of the same color or if one leaf has the same color than its parents, we can apply the quadratic recurrence 🙂

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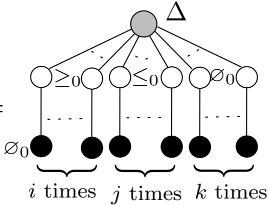
$V^{\Delta}_{i,j,k} :=$  .

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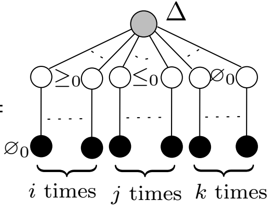

.

By *reversing inequalities*, we obtain relations between the $V^{\Delta}_{i,j,k}$ for various values of i, j and k.

A linear system for long stars

Lemma

For any "rootstock" R, any decoration Δ and any $d \geq 2$, we have

$$\begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 2 & 1 \\ 0 & \dots & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} S(R \mid V^{\Delta}_{1,d-1,0}) \\ \\ \vdots \\ \\ S(R \mid V^{\Delta}_{d-1,1,0}) \end{pmatrix} = \begin{pmatrix} X_1 \\ \\ \vdots \\ \\ X_{d-1} \end{pmatrix} - \begin{pmatrix} S(R|V^{\Delta}_{0,d,0}) \\ 0 \\ \vdots \\ 0 \\ S(R|V^{\Delta}_{d,0,0}) \end{pmatrix},$$

where, for $1 \leq i \leq d-1$,

$$X_i = S(R \mid V^{\Delta}_{i-1,d-1-i,2}) + 2S(R \mid V^{\Delta}_{i-1,d-1-i,1}) \cdot S_{=,0} + S(R \mid V^{\Delta}_{i-1,d-1-i,0}) \cdot S^2_{=,0}.$$

→ this lemma allows to express $S(R \mid V^{\Delta}_{i,j,0})$ in terms of smaller trees since the matrix is invertible!

Thanks for your attention!

D.6. **Trees with 7 vertices.**

D.6.1. *Tree $T_{7,a}$.*

$$
\begin{aligned}
S(T)(\sqrt{t}) &= \frac{5h_1^2+6h_1+(64t+4)\,h_2-(120t+15)}{80t^3} \\
&= 1+6t+44t^2+374t^3+3526t^4+35850t^5+385944t^6+\cdots, \\
S(T)\left(\frac{1}{4}\right) &= \frac{4096}{\pi^2}+\frac{34816}{15\pi}-1152 \approx 1.830035.
\end{aligned}
$$

D.6.2. *Tree $T_{7,b}$.*

$$
\begin{aligned}
S(T)(\sqrt{t}) &= \frac{(40t+54)\,h_1+(176t+11)\,h_2+5h_1^2+10h_1h_2-5h_1^2h_2-(440t+75)}{320t^3} \\
&= 1+6t+45t^2+391t^3+3756t^4+38790t^5+423086t^6+\cdots, \\
S(T)\left(\frac{1}{4}\right) &= -\frac{8192}{3\pi^3}+\frac{7168}{3\pi^2}+\frac{54656}{15\pi}-1312 \approx 1.858234.
\end{aligned}
$$