Random Young diagrams and tableaux

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(based on joint work with J. Borga, C. Boutillier and P.-L. Méliot)

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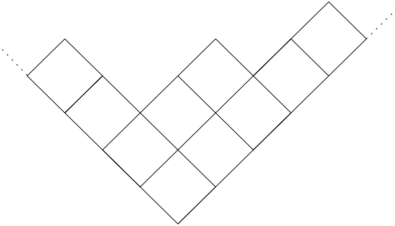
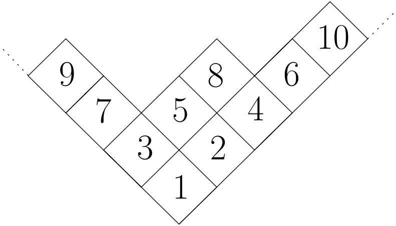
Deuxièmes rencontres LOUCCOUM
Poitiers, novembre 2025

Young diagrams and tableaux



Young diagram

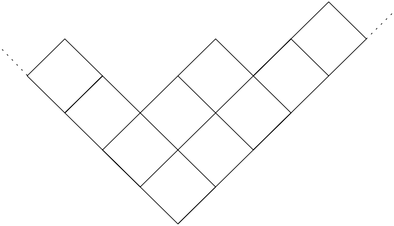


(Standard) Young tableau

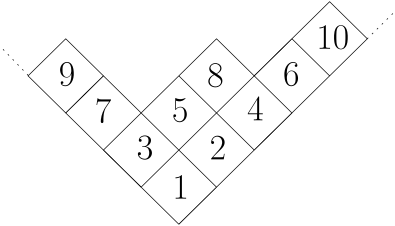
Young diagram: stack of boxes in the upper quarter-plane (encodes an integer partition).

Young tableau: filling of a Young diagram with integers from 1 to n, increasing upwards (encodes a growing sequence of tableaux).

Young diagrams and tableaux

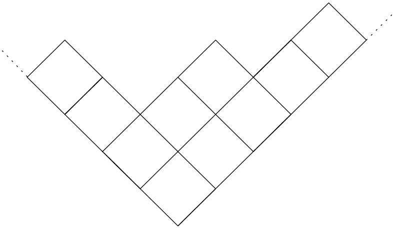


Young diagram

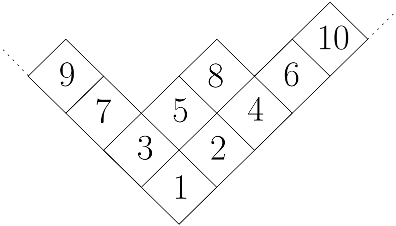


(Standard) Young tableau

Standard object in algebraic combinatorics (symmetric group representation, symmetric functions, . . .)
→ yields tractable models of random walks and random surfaces.



Young diagram



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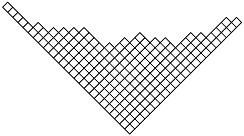
First part: survey some connections with random matrices/random walks
Second part: I'll focus on some recent work with Borga–Boutillier–Méliot on random tableaux.

A well-known connection with random matrices: edge asymptotics

Plancherel measure on diagrams



For a partition λ, we take

$$\mathbb{P}(\lambda) = \frac{\dim(\lambda)^2}{n!}$$

$\dim(\lambda)$: number of tableaux of shape λ.

GUE model of random matrices

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots \\ \overline{a_{1,2}} & \ddots & \vdots \\ \vdots & \cdots & a_{n,n} \end{pmatrix}$$

Hermitian matrix with independent complex Gaussian entries above the diagonal and real Gaussian entries on the diagonal.

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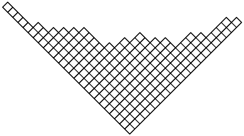

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Hermitian matrix with independent complex Gaussian entries above the diagonal and real Gaussian entries on the diagonal.

Theorem (Borodin–Okounkov–Olshanski, Okounkov, Johansson, ~'00)

Suitably renormalized, for all k, the first rows $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of a random Plancherel Young diagram have the same fluctuations as the largest eigenvalues of a GUE matrix (they both converge to the "Airy ensemble").

Other analogies between random diagrams and random matrices

- Fluctuation of linear statistics ($\sum_{i=1}^n P(\lambda_i)$ for GUE, $\sum_{i=1}^n P(\lambda_i - i)$ for diagrams, where P is a polynomial) are described by similar Gaussian processes (Johansson '98, Kerov–Ivanov–Olshanski '03).

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- "Fixed dimension version" (next slides).

Fixed dimension

Fix an integer $d \geq 1$ and consider a Plancherel random Young diagram **conditioned to have at most d rows**.

Permutation interpretation: we look at the RS shape of a uniform random permutation without decreasing subsequence of length $d+1$.

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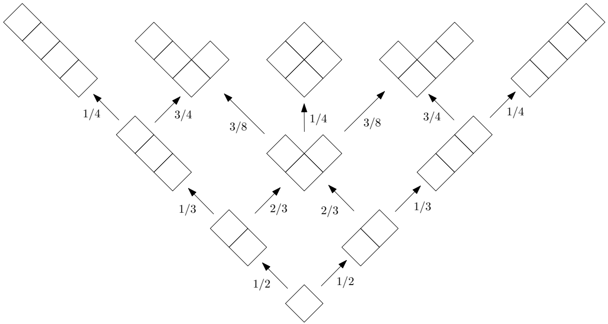
Theorem (Śniady, '06)

Let $\lambda_n = (\lambda_{n,1}, \ldots, \lambda_{n,d})$ be a Plancherel random Young diagram **conditioned to have at most d rows**. *Then*

$$\left(\sqrt{\frac{d}{n}} \, (\lambda_{n,i} - \frac{n}{d}) \right)_{1 \leq i \leq d}$$

converges in distribution to the eigenvalues of a **traceless** *GUE $d \times d$ random matrix.*

Let $\lambda^{(1)}, \dots, \lambda^{(n)}$ be a Markov chain of Young diagrams with $\left|\lambda^{(k)}\right| = k$ and $\mathbb{P}\big(\lambda^{(n)} = \lambda \big| \lambda^{(n-1)} = \mu\big) = \frac{\dim(\lambda)}{n \dim(\mu)}$.



Lemma

For each n, $\lambda^{(n)}$ is Plancherel distributed.

We call the sequence $(\lambda^{(1)}, \dots, \lambda^{(n)})$ a Plancherel random tableau.

Fixed dimension, dynamic version

Fix an integer $d \geq 1$ and consider a Plancherel random tableau $(\lambda^{(1)},\dots,\lambda^{(n)})$ **conditioned to have at most d rows**.

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Theorem (Rizzolo, '19)

$$\left(\sqrt{\frac{d}{n}}\,\big(\lambda_{n,i}^{(tn)} - \frac{nt}{d}\big)\right)_{1\leq i\leq d, 0\leq t\leq 1}$$

converges to a **traceless** *d-dimensional Dyson Brownian motion.*

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Dyson Brownian motion

$\begin{cases} \text{eigenvalues of GUE dynamic matrices (entries are Brownian motions)} \\ \text{universal limit object for random walks in the cone } \{x_1 \geq \cdots \geq x_d\} \end{cases}$

β-deformation (matrix side)

The eigenvalues of GUE random matrices have the following density w.r.t. Lebesgue measure on $\{x_1 \geq x_2 \geq \cdots \geq x_d\}$:

$$\frac{1}{C_d} e^{-(x_1^2+\cdots+x_d^2)} \prod_{i<j} (x_i - x_j)^2.$$

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We **define** GβE ensemble as having the following density w.r.t. Lebesgue measure on $\{x_1 \geq x_2 \geq \cdots \geq x_d\}$:

$$\frac{1}{C_d(\beta)} e^{-\frac{\beta}{2}(x_1^2+\cdots+x_d^2)} \prod_{i<j} (x_i - x_j)^{\beta}.$$

$\beta = 1, 4$: these are eigenvalues of natural models of matrices with real/quaternionic entries.

→ huge literature on this model...

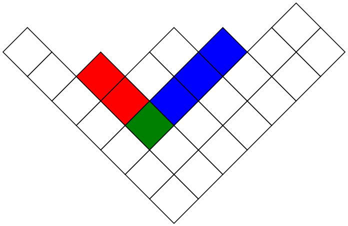
β-deformation (partition side)

The usual Plancherel mesure is defined by

$$\mathbb{P}(\lambda) = \frac{\dim(\lambda)^2}{n!} = \frac{n!}{h_\lambda^2},$$

where

$$h_\lambda = \prod_{(i,j)\in\lambda} \big((\lambda_i - j) + (\lambda'_j - i) + 1\big)$$



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The Jack-Plancherel mesure is defined by

$$\mathbb{P}(\lambda) = \frac{\alpha^n n!}{h_\lambda^{(\alpha)} h_\lambda^{\prime(\alpha)}},$$

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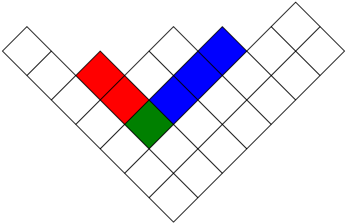

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Note: $\mathbb{P}(\lambda) = [J_\lambda^{(\alpha)}] p_1^n$, where $J_\lambda^{(\alpha)}$ is the (integral) Jack symmetric function indexed by λ. This allows in particular to define a Jack-Plancherel growth process.

Analogies random matrices - random partitions (general β)

- Edge asymptotics of GβE and Jack-Plancherel diagrams are both described by the β-Tracy–Widom distributions (Valkó–Virág, '09, Guionnet–Huang '19).
- Fluctuations of linear statistics are non-centered Gaussian processes (Dimitriu – Edelman, '06, F. – Dołęga, '16).
- Fixed dimension analogies (next slides).

A β version of Śniady's result

Theorem (Matsumoto, '08)

Let $\lambda_n^{(\alpha)} = (\lambda_{n,1}^{(\alpha)}, \dots, \lambda_{n,d}^{(\alpha)})$ *be a Jack-Plancherel random Young diagram* **conditioned to have at most** d **rows**. *Then*

$$\left(\sqrt{\frac{\alpha d}{n}}\ (\lambda_{n,i}^{(\alpha)} - \frac{n}{d}) \right)_{1 \le i \le d}$$

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converges to a d-dimensional **traceless** *GβE ensemble, where* $\beta = 2/\alpha$.

Permutation interpertation: for $\alpha = 2$, $\lambda_{n,1}^{(\alpha)}$ has the same distribution as the LIS of a uniform random fixed-point free involution conditionned to have no decreasing subsequence of length $> 2d$.

A dynamic β version?

Conjecture

Let $\lambda_n^{(\alpha),(1)}, \dots, \lambda_n^{(\alpha),(n)}$ be a Jack–Plancherel random Young tableau **conditioned to have at most** d **rows**. Then

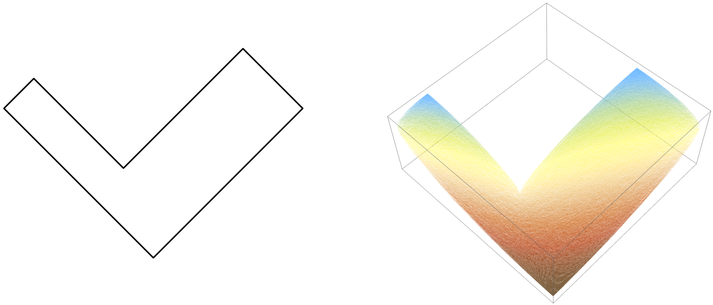
$$\left(\sqrt{\frac{\alpha d}{n}} \, (\lambda_{n,i}^{(\alpha),(tn)} - \frac{tn}{d}) \right)_{1 \le i \le d, 0 \le t \le 1}$$

converges to a to a d-dimensional **traceless** β-Dyson Brownian motion.

Take away message: algebra provides explicit models with interesting asymptotic behaviour!

Interlude

Our model: fix a (large) Young diagram λ (on the left), and take a uniform random Young tableau T of shape λ (on the right).



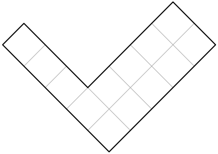
Also studied by Biane, Pittel, Romik, Angel, Holroyd, Virag, Gorin, Rahman, Linusson, Potka, Sulzgruber, Sun, Banderier, Marchal, Wallner, Śniady, Matsumoto, Maślanka, Gordenko, Xu, Prause, Raposo, . . .

Motivations

- Bijection with other models: constrained random permutations (RSK bijection), random sorting networks (Edelman–Greene bijection).
- Asymptotic representation theory: random tableaux encode some asymptotic information on restrictions of representations of large symmetric groups.
- Link with the well-studied lozenge tiling models (Young tableaux are in some sense a limit case of lozenge tilings);
- Tractable model of random linear extensions of 2-dimensional posets.

Simulation (first example)

We consider the n-th dilatation $n \cdot \lambda^0$ of the following diagram



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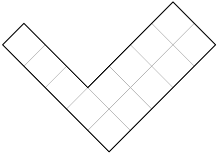

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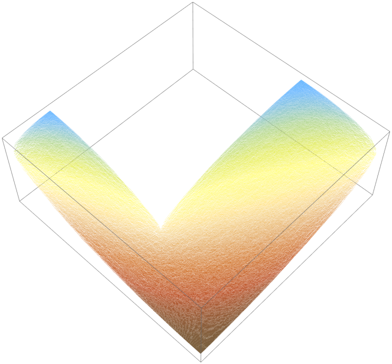
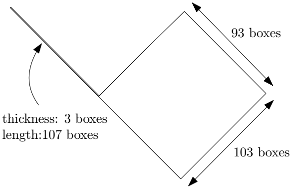
A uniform tableau T_N of shape $n\cdot\lambda^0$:



Here, $n = 100$ so the tableau T_N has $N = 130000$ boxes. There seems to be a smooth limit surface.

Simulation (second example)

This time, take λ^0 to be



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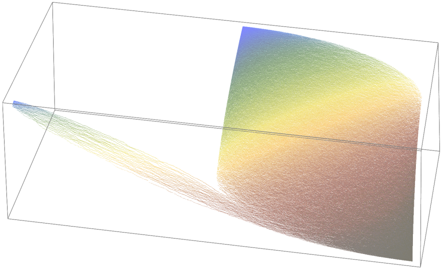
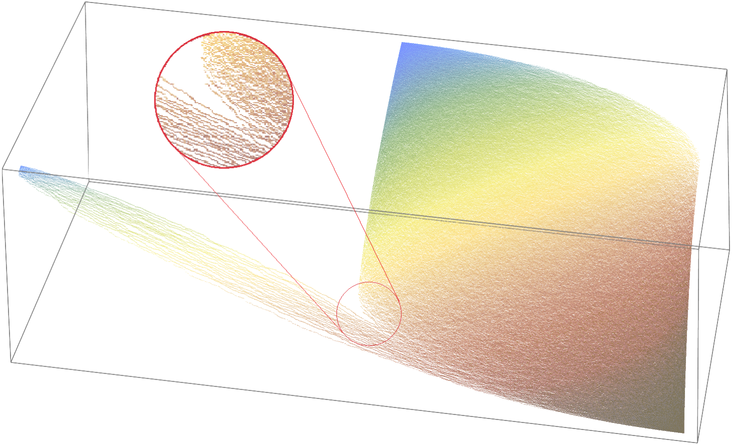


Here, $n = 6$ so the diagram/tableau has $N = 356400$ boxes.

Simulation (second example, with a zoom)



There still seems to be a limiting surface, but this time it is discontinuous!

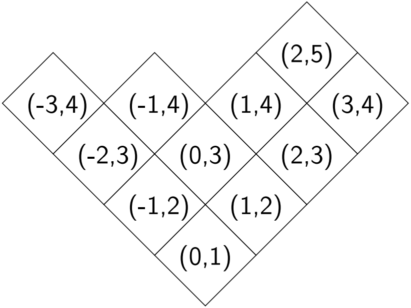
Results (informally)

- Previous contributions (Biane '03, Sun '18): convergence to a limiting surface with some implicit description (via Markov–Krein correspondence and free compression or via a variational principle).
- Our results: a more explicit description of the limit surface in the multirectangular case (dilatation of a fixed diagram λ^0) + characterization of the diagrams λ^0 leading to discontinuous limit surfaces.

Height function

Notation: if T is a tableau of size N, we let

- $T(x,y)$: content of the box with coordinates (x,y) in T;



Box coordinates

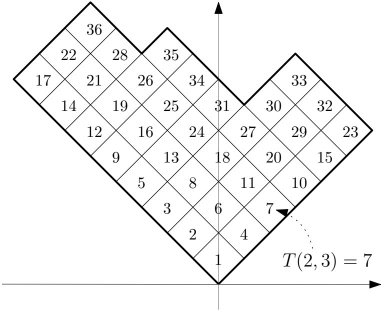


Tableau function

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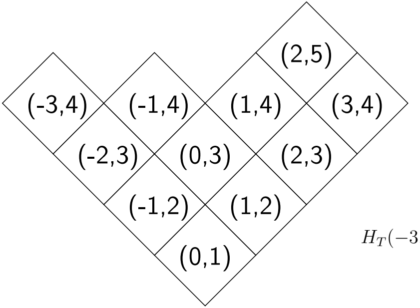
- $T(x,y)$: content of the box with coordinates (x,y) in T;
- $H_T(x,t) = \#\{y : T(x,y) \leq Nt\}$: number of entries on the vertical line x smaller than Nt.



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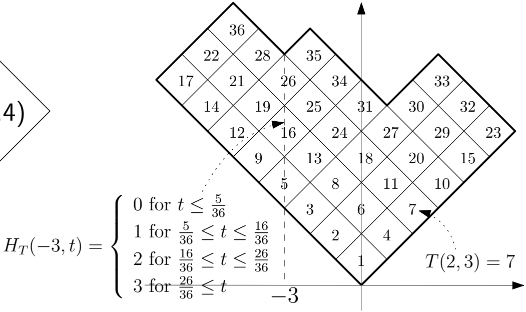

Tableau and height functions

($y \mapsto T(x,y)$ and $t \mapsto H_T(x,t)$ are roughly inverses of each other.)

Existence of the limiting height function

Theorem (Biane '03, Sun '18)

Let λ^0 be a fixed Young diagram. For $n \geq 1$, we let T_N be a uniform random Young tableau of shape $\lambda_N := n \cdot \lambda^0$. Then there exists a deterministic function H^∞ such that

$$\frac{1}{\sqrt{N}} H_{T_N}\left(\lfloor x\sqrt{N} \rfloor, t\right) \xrightarrow[N \to +\infty]{} H^\infty(x,t),$$

in probability, uniformly on (x,t).

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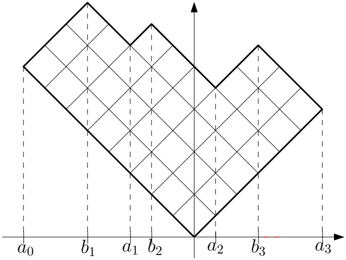
Question: How to compute H^∞?

The critical equation

We encode λ^0 by its interlacing coordinates $a_0 < b_1 < a_1 < \cdots < b_m < a_m$:



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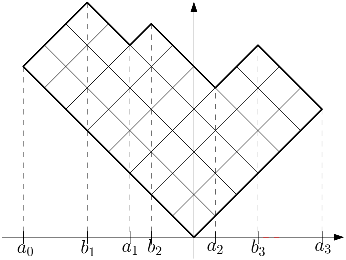

Definition: the critical equation

For parameters (x,t), we consider the polynomial equation

$$U \prod_{i=1}^{m} (x - \eta\, b_i + U) = (1-t) \prod_{i=0}^{m} (x - \eta\, a_i + U),$$

where $\eta = \sqrt{|\lambda^0|}$.

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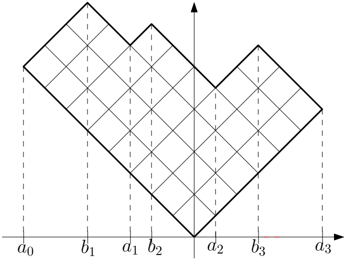

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Lemma

The critical equation has at least $m-1$ real roots.

We denote $U_c(x,t)$ its complex root with positive imaginary part, if it exists (in this case, we say that (x,t) is in the "liquid region").

Formula for the limiting height function

Theorem (Borga, Boutillier, F., Méliot, '23)

$$H^\infty(x,t) = \frac{1}{\pi} \int_0^t \frac{\operatorname{Im} U_c(x,s)}{1-s} \, \mathrm{d}s.$$

Convention: $\operatorname{Im} U_c(x,s) = 0$ if the critical equation has only real root ("frozen region").

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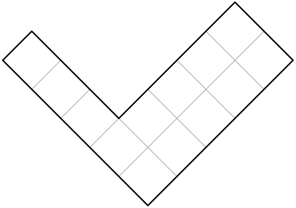
Example: square shape tableaux (Romik–Pittel, '07), $a_0 = -1, b_1 = 0, a_1 = 1$

The critical equation $U(x+U) = (1-t)(x+1+U)(x-1+U)$ is a second degree polynomial equation, and we get

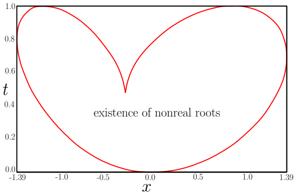
$$H_\diamond^\infty(x,t) = \frac{1}{\pi}\int_0^t \frac{\sqrt{4s-4s^2-x^2}}{2s-2s^2}\,\mathrm{d}s,$$

with the convention that $\sqrt{y} = 0$ if $y \leq 0$.

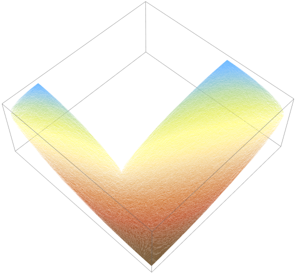
The heart example



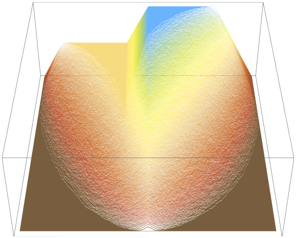
the Young diagram λ^0



boundary of the liquid region

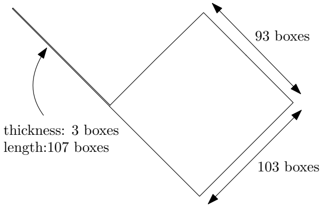


a realization of T_N

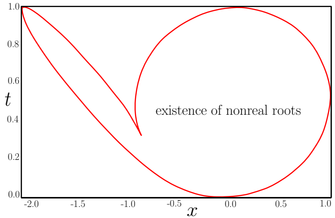


its height function H_{T_N}

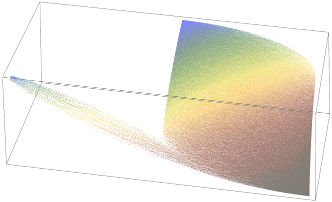
The pipe example



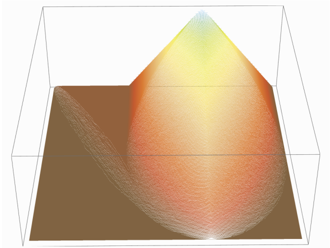
the Young diagram λ^0



boundary of the liquid region

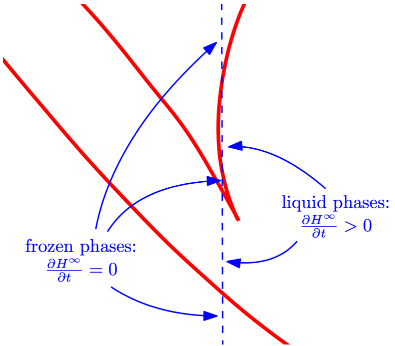


a realization of T_N



its height function H_{T_N}

Why is there a discontinuity in the pipe example?



Zoom on the boundary of the liquid region (blue line $x = x_0 \approx -0.9$)

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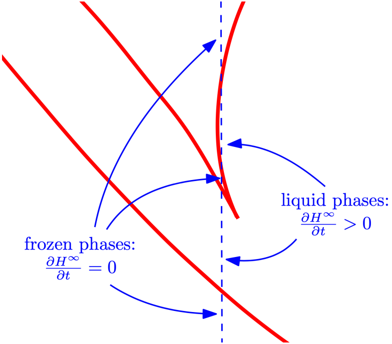

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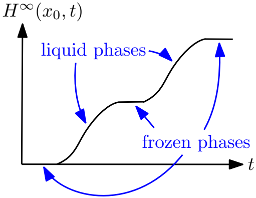

Schematic representation of the function $t \mapsto H^\infty(x_0, t)$

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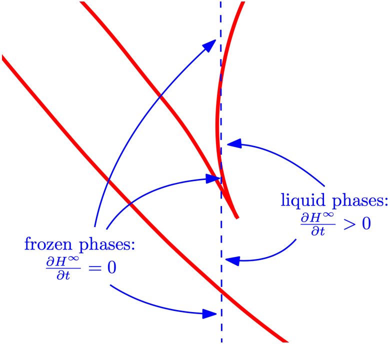

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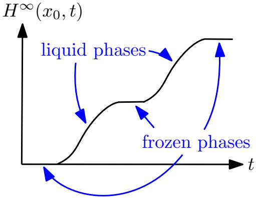

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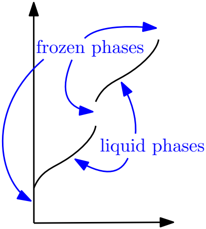
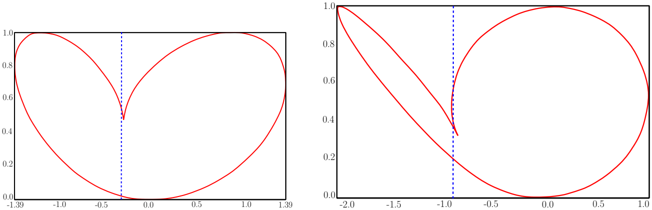


Schematic representation of the function $y \mapsto T^\infty(x_0, y)$

When is there a discontinuity?

There is a discontinuity as soon as the tangent at one of cusp is not vertical (both curves leaving a cusp have the same tangent; think at $x^2 = y^3$).



(In general, there are $m-1$ cusps, where m is the number of distinct parts in λ_0.)

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With some computation, we get

Theorem (Borga, Boutillier, F., Méliot, '23)

The limiting surface $T^{\infty}_{\lambda^0}$ is continuous if and only if the interlacing coordinates $a_0 < b_1 < a_1 < \cdots < b_m < a_m$ of λ^0 satisfy

$$\sum_{\substack{i=0\\ i\neq i_0}}^{m} \frac{1}{a_{i_0} - a_i} = \sum_{i=1}^{m} \frac{1}{a_{i_0} - b_i}, \qquad \text{for all } i_0 = 1, \dots, m-1.$$

In particular, for $m > 1$, the limit surfaces are generically discontinuous!

Proof strategy 1 – determinantal point processes

Notation:
E: locally compact Polish space
μ: reference measure on E
K: measurable function $E^2 \to \mathbb{C}$.
X: simple point process on E

Definition (determinantal point process)

X is a determinantal point process on E with kernel K if it has a joint intensity with respect to μ given by

$$\rho_n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{1 \le i,j \le n},$$

for every $n \ge 1$ and distinct $x_1, \dots, x_n \in E$.

Used a lot in integrable probability theory/statistical physics since 90's, but also in random matrix theory, statistics, ...

Proof strategy 2 – tableaux and bead configurations

Definition (Poissonized tableaux)

A Poissonized tableau of shape λ is an upward increasing filling of λ with real numbers in $[0,1]$.

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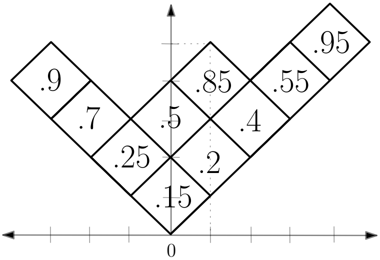
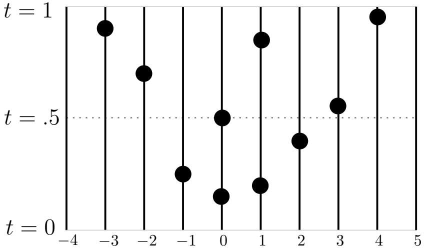
With a Poissonized tableau T, we associate a bead configuration

$$M_T := \Big\{(x, T(x,y)), (x,y) \in \lambda\Big\} \subseteq \mathbb{Z} \times [0,1].$$





Note: $H_T(x,t)$ is the number of beads in $\{x\} \times [0,t]$.

Proof strategy 3 – Gorin–Rahman theorem

Theorem (Gorin, Rahman, '19)

Let T be a uniform random Poissonized tableau of fixed shape λ. Then its associated bead process M_T is a determinantal point process on $\mathbb{Z} \times [0,1]$ with correlation kernel

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Let T be a uniform random Poissonized tableau of fixed shape λ. Then its associated bead process M_T is a determinantal point process on $\mathbb{Z} \times [0,1]$ with correlation kernel

$$K_\lambda((x_1,t_1),(x_2,t_2)) = -\frac{1}{(2\mathrm{i}\pi)^2} \cdot \oint_{\gamma_z}\oint_{\gamma_w} \frac{F_\lambda(z)}{F_\lambda(w)} \frac{\Gamma(w-x_1+1)}{\Gamma(z-x_2+1)} \frac{(1-t_2)^{z-x_2}(1-t_1)^{-w+x_1-1}}{z-w} \mathrm{d}w\,\mathrm{d}z,$$

where $F_\lambda(u) = \Gamma(u+1)\prod_{i=1}^{\infty} \frac{u+i}{u-\lambda_i+i}$ and the double contour integral runs over counterclockwise paths γ_w and γ_z such that

- *γ_w is inside (resp. outside) γ_z if $t_1 \geq t_2$ (resp. $t_1 < t_2$);*
- *γ_w and γ_z contain all the integers in $[-\ell(\lambda), x_1-1]$ and in $[x_2, \lambda_1 - 1]$ respectively;*
- *the ratio $\frac{1}{z-w}$ remains uniformly bounded.*

Proof strategy 4 – Rewriting the kernel

Consequence of Gorin–Rahman's formula:

$$\mathbb{E}[H_T(x,t)] = \int_0^t K((x,s),(x,s))ds$$

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To compute $\lim_{N\to+\infty} \frac{1}{\sqrt{N}} H_{T_N}\left(\lfloor x\sqrt{N}\rfloor, t\right)$, we look for a limit of

$$\frac{1}{\sqrt{N}} K((\lfloor x\sqrt{N}\rfloor, s),(\lfloor x\sqrt{N}\rfloor, s)).$$

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Via Stirling approximation and standard calculus, we get

$$\frac{1}{\sqrt{N}} K\big((\lfloor x\sqrt{N}\rfloor, s),(\lfloor x\sqrt{N}\rfloor, s)\big) \approx -\frac{1}{(2\mathrm{i}\pi)^2}.$$

$$\oint_{\gamma_Z}\oint_{\gamma_W} e^{\sqrt{N}(S(W)-S(Z))}\,\frac{h(W,Z)}{W-Z}\,\mathrm{d}W\,\mathrm{d}Z,$$

where

$S(U) = g(U) - U\log(1-t_0) - \sum_{i=0}^m g(x_0 - \eta\, a_i + U) + \sum_{i=1}^m g(x_0 - \eta\, b_i + U)$

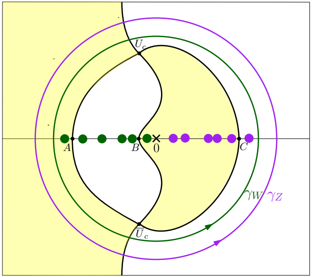
with $g(U) = U\log(U)$ and some function h.

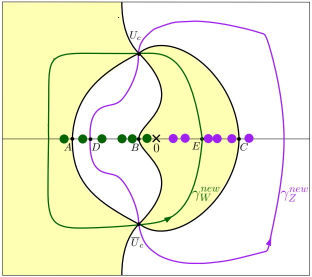
Proof strategy 5 – Steepest descent analysis

Reminder: we are interested in

$$\oint_{\gamma_Z}\oint_{\gamma_W} e^{\sqrt{N}(S(W)-S(Z))}\,\frac{h(W,Z)}{W-Z}\,\mathrm{d}W\,\mathrm{d}Z.$$

Idea: deform γ_Z and γ_W such that $\mathrm{Re}(S(W)) < \mathrm{Re}(S(Z))$ on the new contours.





Schematic representation of the integration contours before and after transformation: in the white (resp. yellow) regions, we have $\mathrm{Re}(S(Z)) > \mathrm{Re}(S(U_c))$ (resp. $\mathrm{Re}(S(Z)) < \mathrm{Re}(S(U_c))$).

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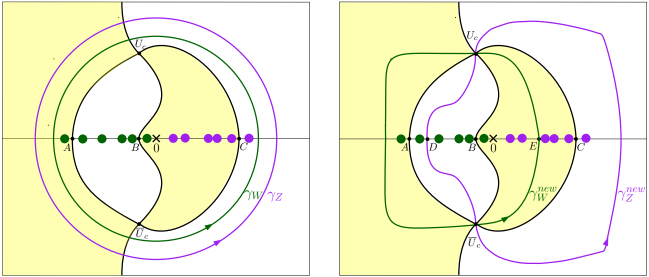

The point U_c on the above picture should satisfy $S'(U_c) = 0$, which is exactly the critical equation! (So the above picture is valid in the liquid region only.)

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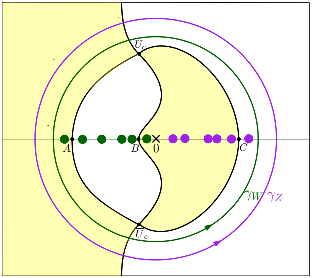

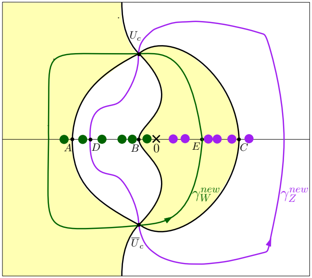

After change of contour, the integral tends to 0. The dominant term asymptotically is the residue term for the pole $W-Z$, which is an integral from $\overline{U_c}$ to U_c.

Thanks for listening!

Commercials:

1. LOUCCOUM Research School (Large Objects Under Combinatorial Constraints and Outside Uniform Model) at CIRM, June 8-12, 2026.

Mini-courses:

- Jean-François Marckert (Bordeaux): Stochastic geometry with combinatorial glasses;
- Sumit Mukherjee (Columbia): Permutons in Statistics;
- Fiona Skerman (Upsalla): Learning on random graphs,

and long talks by Eva-Maria Heinzl and Lucas Teyssier.