# Formes limites de tableaux de Young aléatoires et discontinuités

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Colloquinte de l'équipe PS
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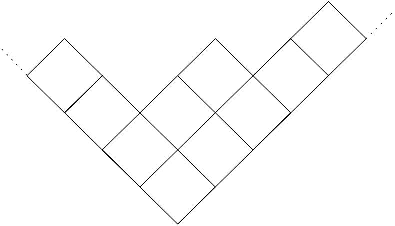
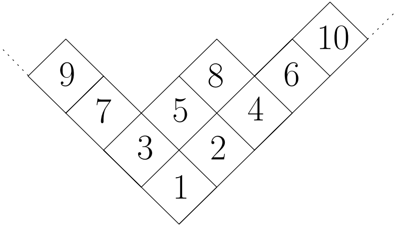
# Young diagrams and tableaux



Young diagram

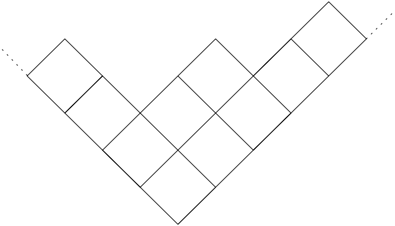



Young tableau

Young diagram: stack of boxes in the upper quarter-plane.

Young tableau: filling of a Young diagram with integers from 1 to $n$, increasing upwards.

# Young diagrams and tableaux



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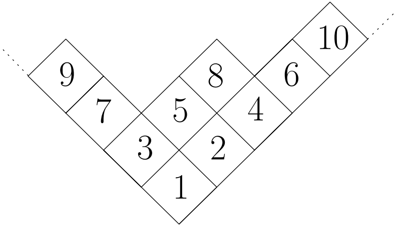



Young tableau

Young diagram: stack of boxes in the upper quarter-plane.

Young tableau: filling of a Young diagram with integers from 1 to $n$, increasing upwards.

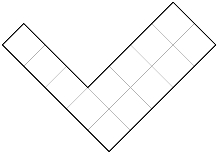
Our model: fix a Young diagram $\lambda$, and take a uniform random Young tableau $T$ of shape $\lambda$ (Biane, Pittel, Romik, Angel, Holroyd, Virag, Gorin, Rahman, Linusson, Potka, Sulzgruber, Sun, Banderier, Marchal, Wallner, Śniady, Matsumoto, Maślanka, Gordenko, Xu, Prause, . . . ).

# Motivations

- Bijection with other models: constrained random permutations (RSK bijection), random sorting networks (Edelman–Greene bijection).
- Asymptotic representation theory: random tableaux encode some asymptotic information on inductions of representations of large symmetric groups.
- Link with the well-studied lozenge tiling models (Young tableaux are in some sense a limit case of lozenge tilings);
- Tractable model of random linear extensions of 2-dimensional posets.

# Simulation (first example)

We consider the $n$-th dilatation $n \cdot \lambda^0$ of the following diagram



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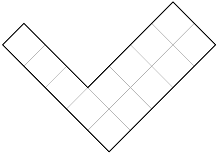
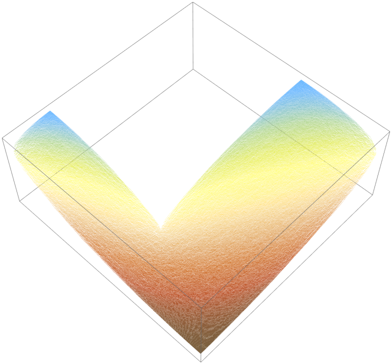


i.e. we replace each box by a $n \times n$ square of boxes.

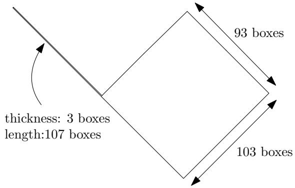
A uniform tableau $T_N$ of shape $n\cdot\lambda^0$:



Here, $n = 100$ so the tableau $T_N$ has $N = 130000$ boxes. There seems to be a smooth limit surface.

# Simulation (second example)

This time, take $\lambda^0$ to be

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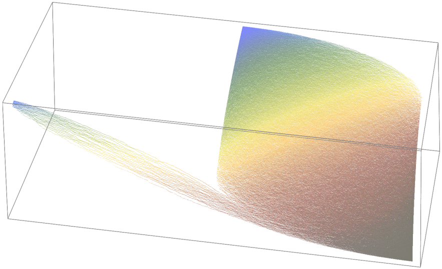
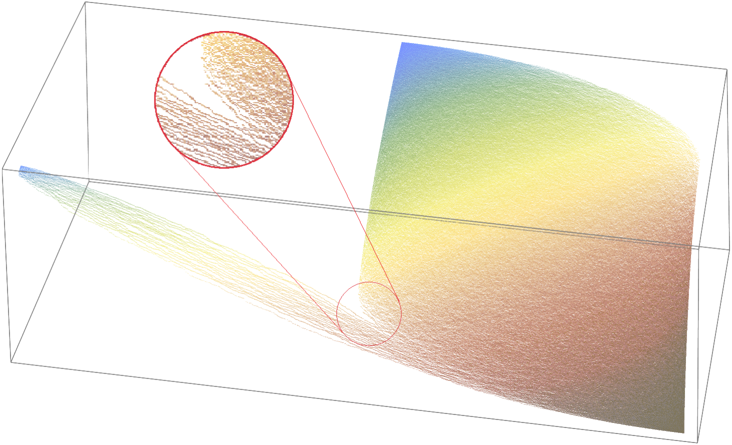


Here, $n = 6$ so the diagram/tableau has $N = 356400$ boxes.

# Simulation (second example, with a zoom)



There still seems to be a limiting surface, but this time it is discontinuous!

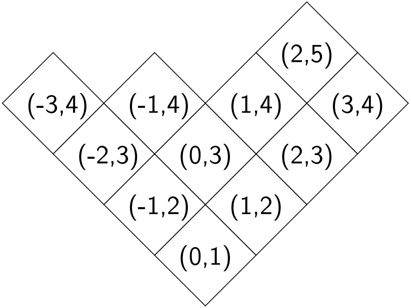
# Results (informally)

- Previous contributions (Biane '03, Sun '18): convergence to a limiting surface with some implicit description (via Markov–Krein correspondence and free compression or via a variational principle).
- Our results: a more explicit description of the limit surface in the multirectangular case (dilatation of a fixed diagram $\lambda^0$) + characterization of the diagrams $\lambda^0$ leading to discontinuous limit surfaces.

# Height function

Notation: if $T$ is a tableau of size $N$, we let

- $T(x,y)$: content of the box with coordinates $(x,y)$ in $T$;




Box coordinates

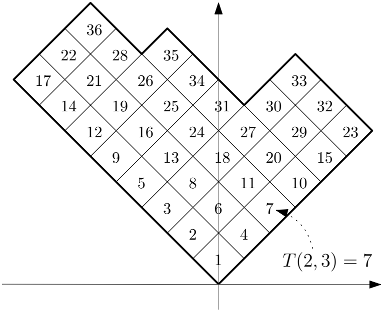
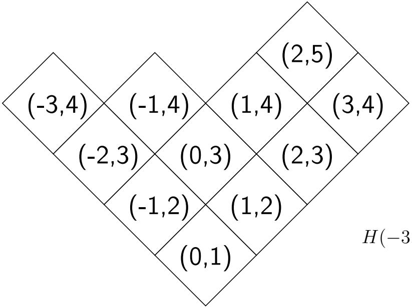



Tableau function

# Height function

Notation: if $T$ is a tableau of size $N$, we let

- $T(x,y)$: content of the box with coordinates $(x,y)$ in $T$;
- $H_T(x,t) = \#\{y : T(x,y) \leq Nt\}$: number of entries on the vertical line $x$ smaller than $Nt$.

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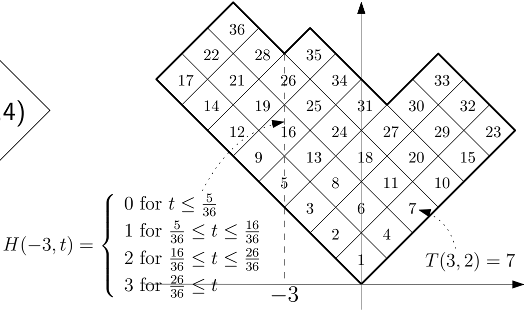


Tableau and height functions

($y \mapsto T(x,y)$ and $t \mapsto H_T(x,t)$ are roughly inverses of each other.)

# Existence of the limiting height function

**Theorem (Biane '03, Sun '18)**

*Let $\lambda^0$ be a fixed Young diagram. For $n \geq 1$, we let $T_N$ be a uniform random Young tableau of shape $\lambda_N := n \cdot \lambda^0$. Then there exists a deterministic function $H^\infty$ such that*

$$\frac{1}{\sqrt{N}} H_{T_N}\left(\lfloor x\sqrt{N} \rfloor, t\right) \xrightarrow[N\to+\infty]{} H^\infty(x,t),$$

*in probability, uniformly on $(x,t)$.*

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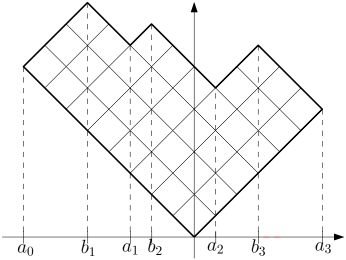
Question: How to compute $H^\infty$?

# The critical equation

We encode $\lambda^0$ by its interlacing coordinates $a_0 < b_1 < a_1 < \cdots < b_m < a_m$:



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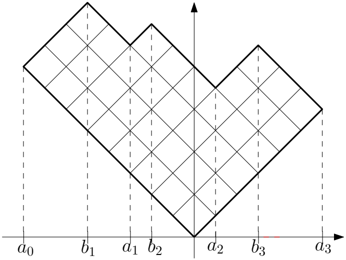


**Definition: the critical equation**

For parameters $(x, t)$, we consider the polynomial equation

$$U \prod_{i=1}^{m} (x - \eta\, b_i + U) = (1-t) \prod_{i=0}^{m} (x - \eta\, a_i + U),$$

where $\eta = \sqrt{|\lambda^0|}$.

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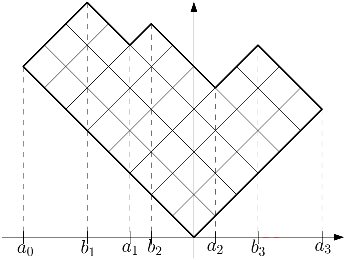


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where $\eta = \sqrt{|\lambda^0|}$.

**Lemma**

*The critical equation has at least $m-1$ real roots.*

We denote $U_c(x,t)$ its complex root with positive imaginary part, if it exists (we say that $(x,t)$ is in the "liquid region").

# Formula for the limiting height function

Theorem (Borga, Boutillier, F., Méliot, '23)

$$H^{\infty}(x,t) = \frac{1}{\pi} \int_0^t \frac{\operatorname{Im} U_c(x,s)}{1-s} \, \mathrm{d}s.$$

Convention: $\operatorname{Im} U_c(x,s) = 0$ if the critical equation has only real root ("frozen region").

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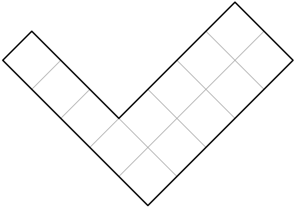
Example: square shape tableaux (Romik–Pittel, '07), $a_0 = -1, b_1 = 0, a_1 = 1$

The critical equation $U(x+U) = (1-t)(x+1+U)(x-1+U)$ is a second degree polynomial equation, and we get

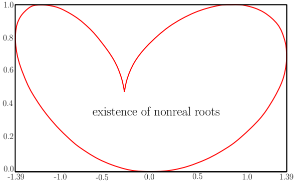
$$H^\infty_\diamond(x,t) = \frac{1}{\pi}\int_0^t \frac{\sqrt{4s-4s^2-x^2}}{2s-2s^2}\,\mathrm{d}s,$$

with the convention that $\sqrt{y} = 0$ if $y \le 0$.

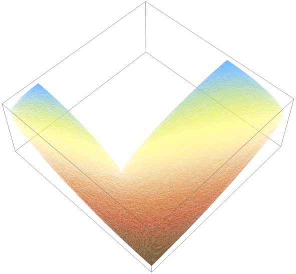
# The heart example



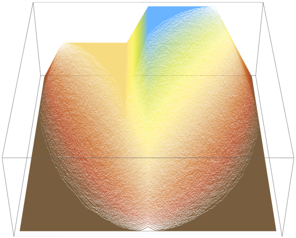
the Young diagram $\lambda^0$




boundary of the liquid region

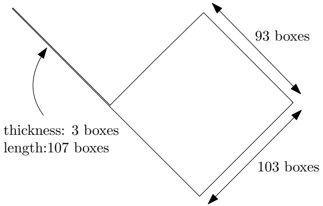


a realization of $T_N$



its height function $H_{T_N}$

# The pipe example

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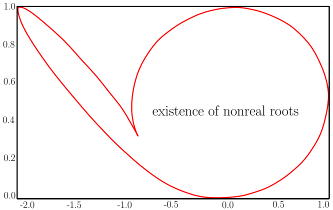


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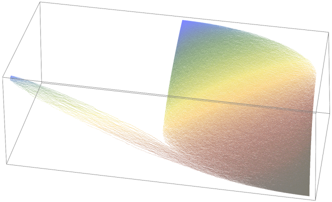

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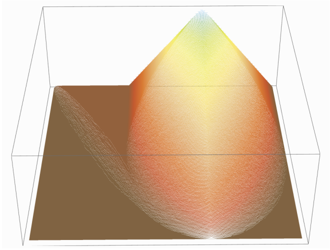
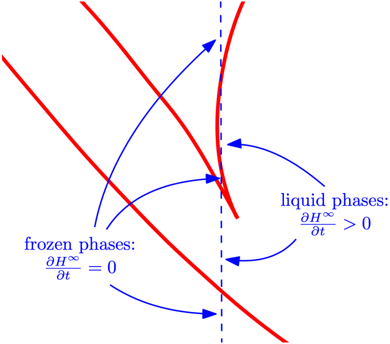


its height function $H_{T_N}$

# Why is there a discontinuity in the pipe example?




Zoom on the boundary of the liquid region (blue line $x = x_0 \approx -0.9$)

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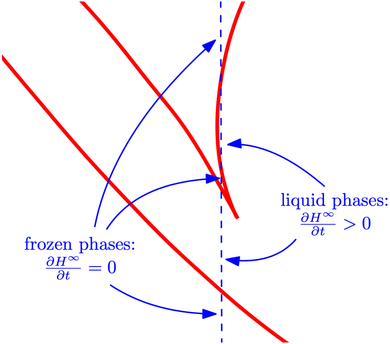


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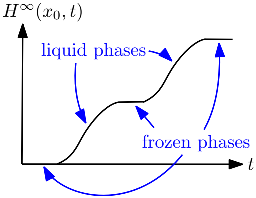


Schematic representation of the function $t \mapsto H^\infty(x_0, t)$

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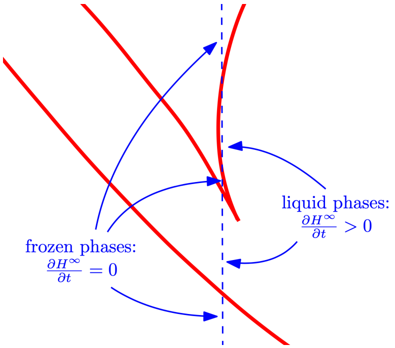


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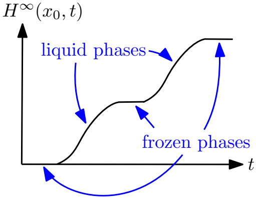


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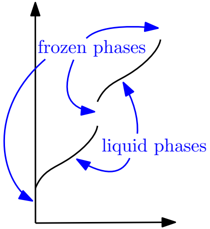
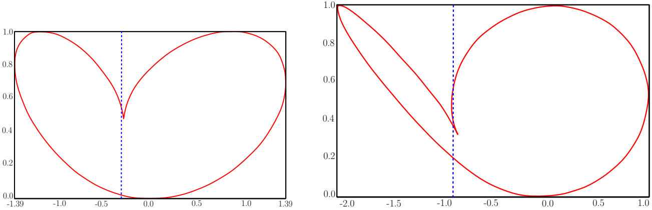



Schematic representation of the function $y \mapsto T^\infty(x_0, y)$

# When is there a discontinuity?

There is a discontinuity as soon as the tangent at one of cusp is not vertical (both curves leaving a cusp have the same tangent; think at $x^2 = y^3$).




(In general, there are $m-1$ cusps, where $m$ is the number of distinct parts in $\lambda_0$.)

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With some computation, we get

**Theorem (Borga, Boutillier, F., Méliot, '23)**

*The limiting surface $T^{\infty}_{\lambda_0}$ is continuous if and only if its interlacing coordinates $a_0 < b_1 < a_1 < \cdots < b_m < a_m$ satisfy*

$$\sum_{\substack{i=0 \\ i \neq i_0}}^{m} \frac{1}{a_{i_0} - a_i} = \sum_{i=1}^{m} \frac{1}{a_{i_0} - b_i}, \qquad \text{for all } i_0 = 1, \dots, m-1.$$

In particular, for $m > 1$, the limit surfaces are generically discontinuous!

## Proof strategy 1 – determinantal point processes

Notation:
$E$: locally compact Polish space
$\mu$: reference measure on $E$
$K$: measurable function $E^2 \to \mathbb{C}$.
$X$: simple point process on $E$

**Definition (determinantal point process)**

$X$ is a determinantal point process on $E$ with kernel $K$ if it has a joint intensity with respect to $\mu$ given by

$$\rho_n(x_1, \ldots, x_n) = \det[K(x_i, x_j)]_{1 \le i,j \le n},$$

for every $n \ge 1$ and distinct $x_1, \ldots, x_n \in E$.

Used a lot in integrable probability theory/statistical physics since 90's, but also in random matrix theory, statistics, ...

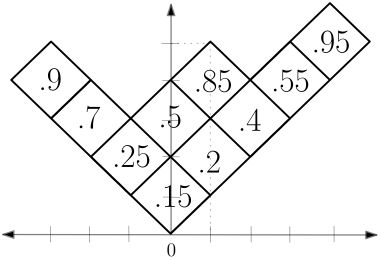
# Proof strategy 2 – tableaux and bead configurations

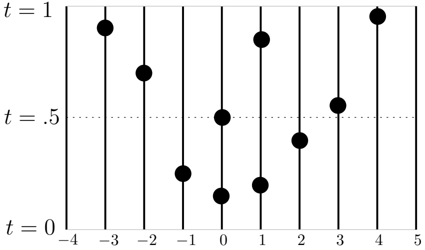
**Definition (Poissonized tableaux)**

A Poissonized tableau of shape $\lambda$ is an upward increasing filling of $\lambda$ with real numbers in $[0,1]$.

With a Poissonized tableau $T$, we associate a bead configuration

$$M_T := \Big\{ (x, T(x,y)), (x,y) \in \lambda \Big\} \subseteq \mathbb{Z} \times [0,1].$$



# Proof strategy 3 – Gorin–Rahman theorem

Theorem (Gorin, Rahman, '19)

*Let $T$ be a uniform random Poissonized tableau of shape $\lambda$. Then its associated bead process $M_T$ is a determinantal point process on $\mathbb{Z} \times [0,1]$ with correlation kernel*

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$$K_\lambda((x_1,t_1),(x_2,t_2)) = -\frac{1}{(2\mathrm{i}\pi)^2} \cdot \oint_{\gamma_z}\oint_{\gamma_w} \frac{F_\lambda(z)}{F_\lambda(w)} \frac{\Gamma(w-x_1+1)}{\Gamma(z-x_2+1)} \frac{(1-t_2)^{z-x_2}(1-t_1)^{-w+x_1-1}}{z-w} \,\mathrm{d}w\,\mathrm{d}z,$$

*where $F_\lambda(u) = \Gamma(u+1)\prod_{i=1}^{\infty} \frac{u+i}{u-\lambda_i+i}$ and the double contour integral runs over counterclockwise paths $\gamma_w$ and $\gamma_z$ such that*

- *$\gamma_w$ is inside (resp. outside) $\gamma_z$ if $t_1 \geq t_2$ (resp. $t_1 < t_2$);*
- *$\gamma_w$ and $\gamma_z$ contain all the integers in $[-\ell(\lambda), x_1-1]$ and in $[x_2, \lambda_1-1]$ respectively;*
- *the ratio $\frac{1}{z-w}$ remains uniformly bounded.*

# Proof strategy 4 – Rewriting the kernel

Consequence of Gorin–Rahman's formula:

$$\mathbb{E}\big[H_T(\lfloor x \rfloor, t)\big] = \int_0^t K((x,s),(x,s))ds$$

## Proof strategy 4 – Rewriting the kernel

Consequence of Gorin–Rahman's formula:

$$\mathbb{E}\big[H_T(\lfloor x \rfloor, t)\big] = \int_0^t K((x,s),(x,s))ds$$

To compute $\lim_{N\to+\infty} \frac{1}{\sqrt{N}} H_{T_N}\left(\lfloor x\sqrt{N}\rfloor, t\right)$, we look for a limit of

$$\frac{1}{\sqrt{N}} K((\lfloor x\sqrt{N}\rfloor, s),(\lfloor x\sqrt{N}\rfloor, s)).$$

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$$\frac{1}{\sqrt{N}} K((\lfloor x\sqrt{N}\rfloor, s),(\lfloor x\sqrt{N}\rfloor, s)).$$

Via Stirling approximation and standard calculus, we get

$$\frac{1}{\sqrt{N}} K\big((\lfloor x\sqrt{N}\rfloor, s),(\lfloor x\sqrt{N}\rfloor, s)\big) \approx -\frac{1}{(2\mathrm{i}\pi)^2}\cdot \oint_{\gamma_Z}\oint_{\gamma_W} e^{\sqrt{N}(S(W)-S(Z))}\,\frac{h(W,Z)}{W-Z}\,\mathrm{d}W\,\mathrm{d}Z,$$

where

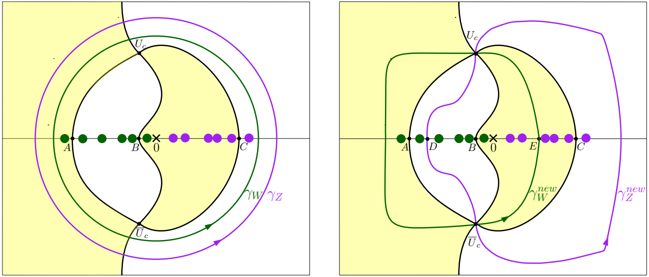
$S(U) = g(U) - U\log(1-t_0) - \sum_{i=0}^m g(x_0 - \eta\, a_i + U) + \sum_{i=1}^m g(x_0 - \eta\, b_i + U)$
with $g(U) = U\log(U)$ and some function $h$.

# Proof strategy 5 – Steepest descent analysis

Reminder: we are interested in

$$\oint_{\gamma_Z} \oint_{\gamma_W} e^{\sqrt{N}(S(W)-S(Z))} \frac{h(W,Z)}{W-Z} \, \mathrm{d}W \, \mathrm{d}Z .$$

Idea: deform $\gamma_Z$ and $\gamma_W$ such that $S(W) < S(Z)$ on the new contours.




Schematic representation of the integration contours before and after transformation: in the white (resp. yellow) regions, we have $S(Z) > S(W)$ (resp. $S(Z) < S(W)$).

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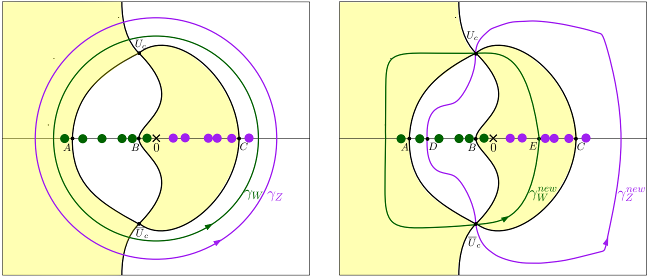


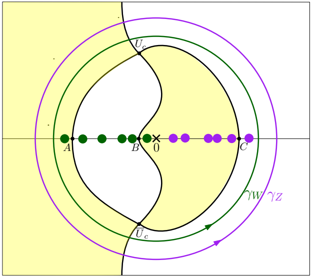

The point $U_c$ on the above picture should satisfy $S'(U_c)=0$, which is exactly the critical equation! (So the above picture is valid in the liquid region only.)

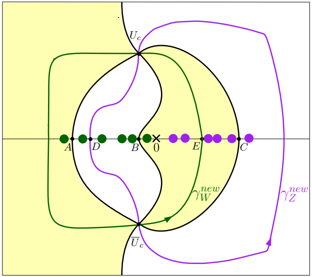
## Proof strategy 5 – Steepest descent analysis

Reminder: we are interested in

$$\oint_{\gamma_Z}\oint_{\gamma_W} e^{\sqrt{N}(S(W)-S(Z))}\,\frac{h(W,Z)}{W-Z}\,\mathrm{d}W\,\mathrm{d}Z\,.$$

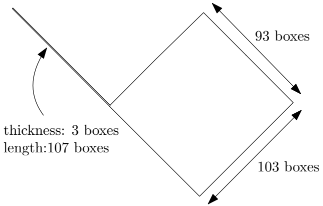
Idea: deform $\gamma_Z$ and $\gamma_W$ such that $S(W) < S(Z)$ on the new contours.



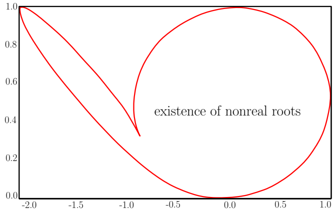




After change of contour, the integral tends to 0. The dominant term asymptotically is the residue term for the pole $W-Z$, which is an integral from $\overline{U_c}$ to $U_c$.

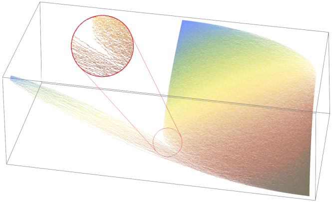
# Merci pour votre attention !



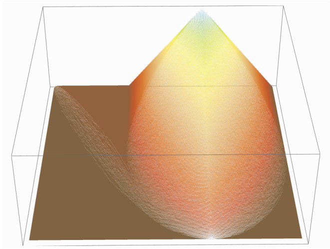

the Young diagram $\lambda^0$




boundary of the liquid region



a realization of $T_N$



its height function $H_{T_N}$