Cumulants mixtes et arbres couvrants

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A problem in random graphs

Erdös-Rényi model of random graphs $G(n, p)$:
- $G$ has $n$ vertices labelled $1, \ldots, n$;
- each edge $(i, j)$ is taken independently with probability $p$;

Example: $n = 8$, $p = 1/2$
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Fix $p \in ]0; 1[$.
Describe asymptotically the fluctuations of the number $T_n$ of triangles.
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Question

Fix $p \in ]0; 1[$. Describe asymptotically the fluctuations of the number $T_n$ of triangles.

Answer (Ruciński, 1988)

The fluctuations are asymptotically Gaussian.
A good tool for that: mixed cumulants

- the $r$-th mixed cumulant $k_r$ of $r$ random variables is a $r$-linear symmetric polynomial in joint moments. Examples:

$$
\kappa_1(X) = \mathbb{E}(X), \quad \kappa_2(X, Y) = \text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \\
\kappa_3(X, Y, Z) = \mathbb{E}(XYZ) - \mathbb{E}(XY)\mathbb{E}(Z) - \mathbb{E}(XZ)\mathbb{E}(Y) \\
- \mathbb{E}(YZ)\mathbb{E}(X) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z).
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- if the variables can be split in two mutually independent sets, then the cumulant vanishes.

- if, for each $r \neq 2$, the sequence $\kappa_r(X_n, \ldots, X_n)$ converges towards 0 and if $\text{Var}(X_n)$ has a limit, then $X_n$ converges in distribution towards a Gaussian law.
Application to the number of triangles

\[ T_n = \sum_{\Delta = \{i, j, k\} \subset [n]} B_\Delta, \quad \text{where } B_\Delta(G) = \begin{cases} 1 & \text{if } G \text{ contains the triangle } \Delta; \\ 0 & \text{otherwise.} \end{cases} \]
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By multilinearity, \( \kappa_\ell(T_n) = \sum_{\Delta_1, \ldots, \Delta_\ell} \kappa_\ell(B_{\Delta_1}, \ldots, B_{\Delta_\ell}). \)
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Example:

\[ \kappa_\ell(B_{\Delta_1}, \ldots, B_{\Delta_7}) = 0. \]

\( \{\Delta_1, \Delta_2, \Delta_5, \Delta_7\} \text{ is independent from } \{\Delta_3, \Delta_4, \Delta_6\}. \)

Reminder: presence of different edges are independent events.
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Triangles need to share an edge to be dependent!

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But most of the terms vanish (because the variables are independent).

Example:

This configuration contributes to the sum. Call it configuration of dependent triangles. Note that it has only $$\ell + 2$$ vertices (here $$\ell = 8$$).

$$\kappa_\ell(B_{\Delta_1}, \ldots, B_{\Delta_8}) \neq 0.$$
**Bound on the cumulant**

\[ \kappa_\ell(T_n) = \sum_{\Delta_1, \ldots, \Delta_\ell} \kappa_\ell(B_{\Delta_1}, \ldots, B_{\Delta_\ell}). \]

- only configurations of dependent triangles contribute to the sum;
Bound on the cumulant

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- only configurations of dependent triangles contribute to the sum;
- the number of unlabelled configurations of dependent triangles does not depend on \( n \) (only on \( \ell \));
- each configuration can be labelled in at most \( n^{\ell+2} \) ways.

Conclusion

\[ |\kappa_\ell(T_n)| = O_\ell(n^{\ell+2}) \]
The central limit theorem for triangles

Proposition (Leonov, Shirryaev, 1955)
If $X_1, \ldots, X_\ell$ can be split into two sets of mutually independent variables, then

$$\kappa_\ell(X_1, \cdots, X_\ell) = 0$$

Corollary (Janson, 1988)
For each $\ell$, there exists a constant $C_\ell$ such that

$$|\kappa_\ell(T_n)| = C_\ell n^{\ell+2}$$
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Corollary (Ruciński, 1988)
$$\frac{T_n - \mathbb{E}(T_n)}{\sqrt{\text{Var}(T_n)}} \to \mathcal{N}(0, 1)$$

Proof: $\text{Var}(T_n) \approx n^4$ and $\kappa_\ell(T_n/n^2) = n^{2-\ell} = o(1)$ for $\ell > 2$. 
Our work

**Theorem (F., Méliot, Nighekbali, 2014)**

Let \( X_1, \ldots, X_\ell \) be random variables with finite moments of order \( \ell \),

\[
|\kappa_\ell(X_1, \cdots, X_\ell)| \leq 2^{\ell - 1} \|X_1\|_\ell \cdots \|X_\ell\|_\ell \cdot \text{ST} \left( G_{\text{dep}}(X_1, \cdots, X_\ell) \right),
\]

where \( \text{ST} \left( G_{\text{dep}}(X_1, \cdots, X_\ell) \right) \) is the number of spanning trees of the dependency graph of \( X_1, \cdots, X_\ell \).

Dependency graphs for a list \((B_{\Delta_1}, \cdots, B_{\Delta_\ell})\):

\[ B_{\Delta_i} \sim B_{\Delta_j} \iff \Delta_i \text{ and } \Delta_j \text{ share an edge} \]

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Corollary (FMN, 2014)

There exists an absolute constant $C$ such that

$$|\kappa_\ell(T_n)| = (C\ell)^\ell n^{\ell+2}$$

Naive bound: $(C\ell)^{3\ell} n^{\ell+2}$
Our work

**Theorem (F., Méliot, Nighekbali, 2014)**

Let $X_1, \ldots, X_\ell$ be random variables with finite moments of order $\ell$,

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**Corollary (FMN, 2014)**

Very precise extension of the central limit theorem: if $1 \ll v \ll n^{1/2}$,

$$\mathbb{P} \left[ T_n \geq \binom{n}{3} p^3 + v \cdot n^2 \right] \sim \frac{1}{\sqrt{\pi p^5 (1-p)v^2}} \exp \left( - \frac{v^2}{p^5 (1-p)} + \frac{7-8p}{2n p(1-p)^{3/2}} \right)$$
Moment-cumulant relation

Mixed cumulants can be expressed in terms of mixed moments:

\[ \kappa(X_1, \ldots, X_r) = \sum_{\pi} \mu(\pi) M_\pi, \]

where

- \( \pi \) runs over set-partitions of \([\ell]\),
- \( \mu(\pi) = \mu(\pi, \{[\ell]\}) \) is the Möbius function of the set-partition poset (it is explicit!),
- \( M_\pi = \prod_{B \in \pi} \mathbb{E}[\prod_{i \in B} X_i] \).

Example:

\[ \kappa_3(X, Y, Z) = \mathbb{E}(XYZ) - \mathbb{E}(XY)\mathbb{E}(Z) - \mathbb{E}(XZ)\mathbb{E}(Y) \]
\[ - \mathbb{E}(YZ)\mathbb{E}(X) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z). \]
Using independence to simplify $M_\pi$

Example: $\pi = \{\{1, 2, 3, 4\}, \{5, 6\}\}$ and

\[
H := G_{\text{dep}}(X_1, \ldots, X_6) = \begin{array}{c}
\begin{array}{c}
3 \\
2 \\
4 \\
5 \\
6
\end{array}
\end{array}
\]

Then $M_\pi := \mathbb{E}(X_1 X_2 X_3 X_4) \mathbb{E}(X_5 X_6) \]
\[
= \mathbb{E}(X_1 X_2) \mathbb{E}(X_3 X_4) \mathbb{E}(X_5) \mathbb{E}(X_6).
\]

In general, $M_\pi = M_{\phi_H(\pi)}$, with obvious definition of $\phi_H(\pi)$. 
Rewriting the summation

\[ \kappa(X_1, \ldots, X_r) = \sum_{\pi} \mu(\pi) M_\pi = \sum_{\pi} \mu(\pi) M_{\phi_H(\pi)} \]

\[ = \sum_{\pi'} M_{\pi'} \left( \sum_{\pi \text{ s.t. } \phi_H(\pi) = \pi'} \mu(\pi) \right) \]
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- \( \phi_H(\pi) = \pi' \) \implies for all part \( \pi'_i \) of \( \pi' \), the induced graph \( H[\pi'_i] \) is connected.
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- \( \phi_H(\pi) = \pi' \Rightarrow \) for all part \( \pi'_i \) of \( \pi' \), the induced graph \( H[\pi'_i] \) is connected.
- if so, we have to compute

\[ \alpha^{\pi'}_H := \sum_{\pi \text{ s.t. } \phi_H(\pi) = \pi'} \mu(\pi). \]
Consider the contracted graph \( H/\pi \). Example:

\[
\pi = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}
\]

It is a multigraph.
Consider the contracted graph $H/\pi$. Example:

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\pi = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}
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Lemma

\[
\alpha_H^{\pi'} = \sum_{E \subseteq E(H/\pi')} (-1)^{|E|},
\]

where the sum runs over spanning connected subgraphs of $H/\pi'$.

If $H/\pi'$ is connected, $|\alpha_H^{\pi'}|$ is Tutte polynomial evaluated at $(1, 0)$. 
$\alpha^\pi_H$ and Tutte polynomial

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**Corollary:** $|\alpha^\pi_H| \leq ST(H/\pi')$. 

V. Féray (with PLM, AN) (I-Math, UZH)
Cumulants mixtes et arbres couvrants
CIRM, 2014–03
Bounding everything

Reminder:

$$\kappa(X_1, \ldots, X_\ell) = \sum_{\pi'} M_{\pi'} \alpha_{H}^{\pi'}$$

where the sum runs over set-partition $\pi'$ such that the induced graphs $H[\pi']$ are connected.
Bounding everything

Reminder:

\[ \kappa(X_1, \ldots, X_\ell) = \sum_{\pi'} M_{\pi'} \alpha_H^{\pi'} \prod_i 1_{H[\pi'_i]} \text{ connected} \]
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\[ \kappa(X_1, \ldots, X_\ell) = \sum_{\pi'} M_{\pi'} \alpha_{H}^{\pi'} \prod_{i} 1_{H[\pi'_i]} \text{ connected} \]

We have the following inequalities

\[ |M_{\pi}| \leq |X_1|_\ell \cdots |X_\ell|_\ell \] (Hölder inequality);
\[ \left| \alpha_{H}^{\pi'} \right| \leq \text{ST}(H/\pi'); \]
\[ 1_{H[\pi'_i]} \text{ connected} \leq \text{ST}(H[\pi'_i]) \]
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Thus

$$|\kappa(X_1, \ldots, X_\ell)| \leq ||X_1||_\ell \cdots ||X_\ell||_\ell \left[ \sum_{\pi'} \text{ST}(H/\pi') \left( \prod_i \text{ST}(H[\pi'_i]) \right) \right]$$
A combinatorial identity

Lemma

\[ 2^{\ell-1} \text{ST}(H) = \sum_{\pi'} \text{ST}(H/\pi') \left( \prod_i \text{ST}(H[\pi'_i]) \right), \]

where the sum runs over all set-partitions of \([\ell]\).
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A precise bound on cumulants of $T_n$

Recall that $\kappa_\ell(T_n) = \sum_{\Delta_1,\ldots,\Delta_\ell} \kappa_\ell(B_{\Delta_1}, \ldots, B_{\Delta_\ell})$.

Thus

$$|\kappa_\ell(T_n)| \leq \sum_{\Delta_1,\ldots,\Delta_\ell} 2^{\ell-1} \left| \text{ST} \left( G_{\text{dep}}(B_{\Delta_1}, \ldots, \Delta_\ell) \right) \right|.$$
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Fix a Cayley tree. For how many lists of triangles is it contained in $G_{\text{dep}}(B_{\Delta_1}, \ldots, \Delta_\ell)$?
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A new bound on cumulants via spanning trees

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- $\Delta_2$ should have an edge in common with $\Delta_5$. Also $3n - 6$ choices.
- ...
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\[ |\kappa_\ell(T_n)| \leq \ell^{\ell-2} \binom{n}{3}(6n - 12)^{\ell-1} \leq (6\ell)^\ell n^{\ell+2} \]
Moderate deviations

Let \( X_n = (T_n - \mathbb{E}(T_n))/n^{5/3} \), then

\[
\log \mathbb{E}(\exp(zX_n)) = \sum_{\ell \geq 2} \kappa^{(\ell)}(X_n) z^\ell / \ell!
\]

\[
= n^{2/3} \sigma^2 z^2 / 2 + L z^3 / 6 + \sum_{\ell \geq 4} n^{5/3} \kappa^{(\ell)}(T_n) z^\ell / \ell!
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call it \( R \)
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\]

But \(|R| \leq \sum_{\ell \geq 4} n^{2(3-\ell)/3} (C\ell)^\ell z^\ell / \ell! = O(n^{-2/3})\) locally uniformly for \( z \) in \( \mathbb{C} \). Thus

\[
\mathbb{E}(\exp(zX_n)) = \exp \left( n^{2/3} \sigma^2 z^2 / 2 + L z^3 / 6 \right) (1 + O(n^{-2/3})).
\]
Moderate deviations

Let $X_n = (T_n - \mathbb{E}(T_n))/n^{5/3}$, then

$$\log \mathbb{E}(\exp(zX_n)) = \sum_{\ell \geq 2} \kappa^{(\ell)}(X_n)z^{\ell}/\ell!$$

$$= n^{2/3} \sigma^2 z^2/2 + Lz^3/6 + \sum_{\ell \geq 4} n^{5/3} \kappa^{(\ell)}(T_n)z^{\ell}/\ell!$$

But $|R| \leq \sum_{\ell \geq 4} n^{2(3-\ell)/3} (C\ell)^{\ell}z^{\ell}/\ell! = O(n^{-2/3})$ locally uniformly for $z$ in $\mathbb{C}$. Thus

$$\mathbb{E}(\exp(zX_n)) = \exp \left( n^{2/3} \sigma^2 z^2/2 + Lz^3/6 \right) \left( 1 + O(n^{-2/3}) \right).$$

→ looks like, but is stronger than the hypotheses in Hwang’s quasi-power theorem (convergence on $\mathbb{C}!$) ⇒ stronger results.
Conclusion

- very general bound on mixed cumulants, with a strong combinatorial flavor;
- implies a good uniform bound on cumulants of sums of partially dependent random variables (number of copies of subgraphs, character of a random irreducible representation, . . .);
- implies some precise deviation results.
Conclusion

- very general bound on mixed cumulants, with a strong combinatorial flavor;

- implies a good uniform bound on cumulants of sums of partially dependent random variables (number of copies of subgraphs, character of a random irreducible representation, ...);

- implies some precise deviation results.

Questions:

- Large deviations $\mathbb{P}(T_n \geq \mathbb{E}(T_n) + \nu n^3) \sim ?$;

- other models: $p_n \to 0$, $G(n, M)$ (fixed number of edges $\Rightarrow$ almost-independence).