**Part 1: Background and main result**

### b-conjecture I

Goulden and Jackson [3] defined a family of coefficients $h_{\mu,\nu}(\alpha - 1)$ by the following identity:

$$
\log \left( \sum_{\alpha \geq 0} \frac{J_{\alpha}^{(\alpha)}(x) J_{\lambda}^{(\alpha)}(y) J_{\mu}^{(\alpha)}(z) \alpha!}{h_{\lambda}(\alpha) h_{\mu}(\alpha)} \right) = \sum_{\alpha \geq 0} \sum_{\mu,\nu} h_{\mu,\nu}(\alpha - 1) p_{\mu}(x) p_{\nu}(y),
$$

where $h_{\lambda}(\alpha)$ and $h_{\mu}(\lambda)$ are $\alpha$-deformation of the "hook-product". Motivation:

- $h_{\mu,\nu}(0)$ enumerates connected hypergraphs embedded into oriented surfaces with particular statistics given by $\mu$, $\nu$ and $\tau$.
- $h_{\mu,\nu}(1)$ enumerates connected hypergraphs embedded into non-oriented surfaces with the same statistics.

### b-Conjecture II

A previous result of us states that $h_{\mu,\nu}(\beta)$ is a rational function in $\alpha$ with only possible poles at $\alpha = 0$. Our main result is a proof that, in fact, $h_{\mu,\nu}(\beta)$ has no pole at $\alpha = 0$, thus completing the proof of *polynomiality in $b$-Conjecture*.

**Theorem (Main Result)**

For all partitions $\tau, \mu, \nu \vdash n \geq 1$ quantity $h_{\mu,\nu}(\beta)$ is a **polynomial** in $\beta$ of degree $2 + n - \ell(\tau) - \ell(\mu) - \ell(\nu)$ with rational coefficients.

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**Part 2: Strong factorization property**

### Two equivalent SFP

Let $R$-ring, and $u = \{u_{\lambda}\}_{\lambda \subseteq \mathbb{P}}$ family of elements of $R(\alpha)$ indexed by subsets of $[r]$.

**Proposition (F, 2013)**

Assume $u_{\lambda}, u_{\lambda'}^{\perp} = O(1)$. The following are equivalent:

- for all $H \subseteq [r]$, $\prod_{\lambda \in H} u_{\lambda}^{\perp} = 1 + O(|H|^{1/2})$;
- for all $H \subseteq [r]$, $\prod_{\lambda \in H} u_{\lambda}^{\perp} = 1 + O(|H|^{1/4})$.

If this holds, we say that $u$ has the **strong factorization property (SFP)**.

**Corollary:**

For two families $(u_{\lambda})_{\lambda \subseteq \mathbb{P}}$ and $(v_{\lambda})_{\lambda \subseteq \mathbb{P}}$ with SFP, their entry-wise product $(u_{\lambda} v_{\lambda})_{\lambda \subseteq \mathbb{P}}$ and quotient $(u_{\lambda} / v_{\lambda})_{\lambda \subseteq \mathbb{P}}$ also have SFP.

**Back to our main theorem**

**Lemma**

$$
\log \left( \sum_{\alpha \geq 1} \frac{J_{\alpha}^{(\alpha)}(x) J_{\lambda}^{(\alpha)}(y) J_{\mu}^{(\alpha)}(z) \alpha!}{h_{\lambda}(\alpha) h_{\mu}(\alpha)} \right) = \sum_{\alpha \geq 1} \sum_{\mu,\nu} k^{2}(\lambda, \mu, \nu) h_{\lambda}(\alpha) h_{\mu}(\alpha),
$$

where $G(\lambda) = \sum_{\mu,\nu} k^{2}(\lambda, \mu, \nu) J_{\lambda}^{(\alpha)}(x) J_{\mu}^{(\alpha)}(y) J_{\nu}^{(\alpha)}(z)$.

Thus our main theorem is equivalent to:

$$
k^{2}(1^{h}, \ldots, 1^{h}) = O(\alpha^{n}).
$$

For any $\tau \geq 1$ and for any partitions $\lambda', \ldots, \lambda$, the family $u_{\lambda'} := J_{\lambda'}^{(1^{h})}$ has SFP,

- the family $v_{\lambda} := h_{\lambda}(\alpha)$ has SFP,
- the following family also has SFP:
  - $u_{\lambda} := h_{\lambda}(\alpha) = \alpha^{\lambda \cdot (\Pi_{i \geq 1} \lambda^{(i)})^{-1} h_{\lambda}(\alpha)}$.

As a consequence, the family $u_{\lambda} / v_{\lambda} = G(\lambda)$ has SFP.

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**References**

