Permutons

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What are permutons?

- measures which represent limits of large permutations;
  - scaling limits;
  - convergence of densities of substructures.
- questions on large permutations can be translated as questions on measures
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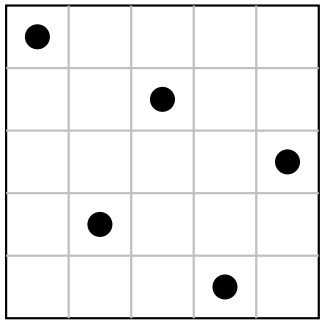
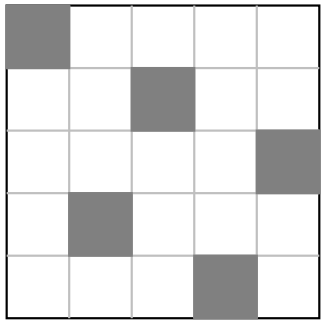
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Fits in general attempts

- to develop limit theory for combinatorial objects: graphons (Lovasz and co-authors), dendrons, ...;
- to understand scaling limits of combinatorial objects (partitions 70's, trees 90's, maps and graphs '00s, '10s, ...).

"Scaling limit" of large permutations

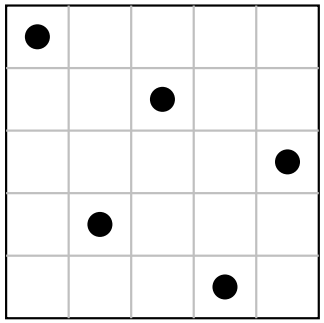
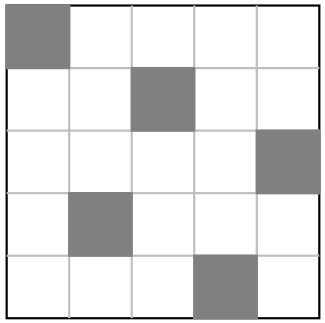
A permutation π can be encoded as a probability measure μ_π on $[0,1]^2$.

$\pi = 52413 =$  $\mapsto \quad \mu_\pi =$ 

In μ_π, each small square has weight $1/n$ (i.e. density n).

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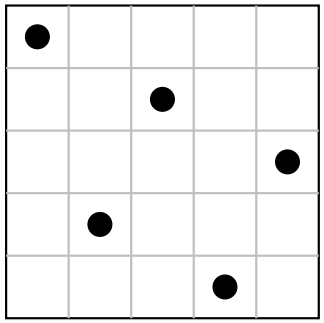
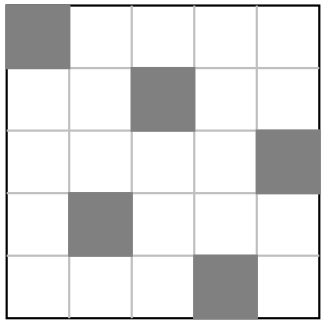
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$\rightarrow$ potential limits also have uniform marginals.

Definition

A permuton is a probability measure on $[0,1]^2$ with uniform marginals.

An example: Mallows permutations

Mallows model on S_n: $\mathbb{P}(\sigma_n) \propto q_n^{\mathrm{inv}(\sigma_n)}$,
where $\mathrm{inv}(\sigma) = \#\{(i,j) \text{ with } i < j \text{ and } \sigma(i) > \sigma(j)\}$.

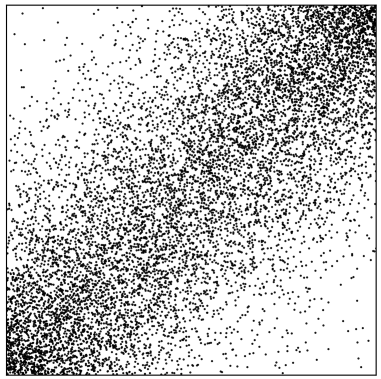
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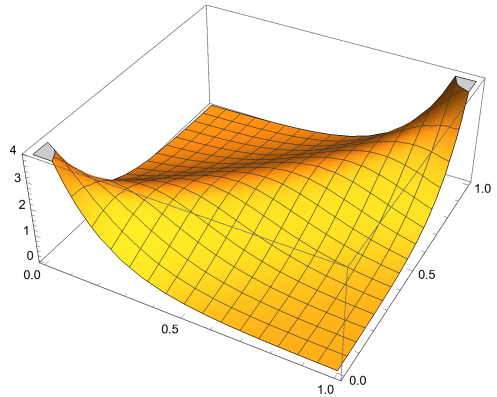
Theorem (Starr, '09)

Take $q_n = 1 - \beta/n$. Then $\mu_{\boldsymbol{\sigma}^{(n)}}$ converge to the deterministic permuton with density

$$u(x,y) = \frac{(\beta/2)\sinh(\beta/2)}{\left(e^{\beta/4}\cosh(\beta[x-y]/2) - e^{-\beta/4}\cosh(\beta[x+y-1]/2)\right)^2}.$$



Simulation ($n = 10000$, $\beta = 6$)



$\beta = 6$

Pattern densities

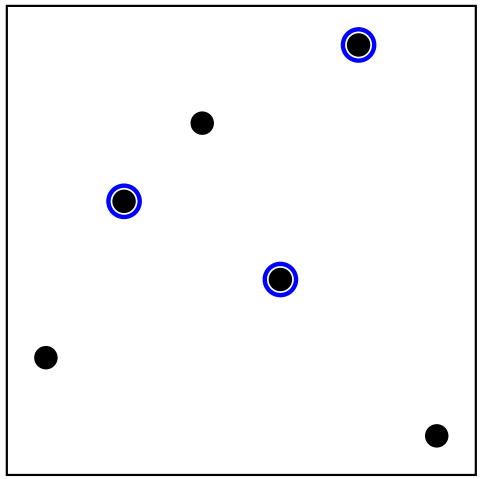
Definition: An occurrence of a pattern τ in σ is a subsequence $\sigma_{i_1} \dots \sigma_{i_k}$ that is order-isomorphic to τ, *i.e.* $\sigma_{i_s} < \sigma_{i_t} \Leftrightarrow \tau_s < \tau_t$.

Example (occurrences of 2 1 3)

2 4 5 3 6 1

8 2 3 4 6 1 7 5

Visual interpretation



Pattern densities

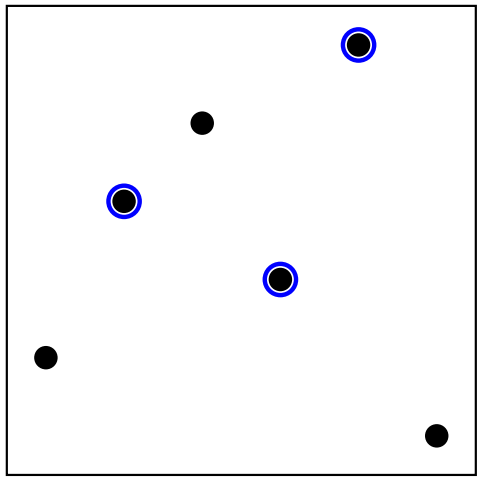
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Visual interpretation



If τ and σ are permutations of size k and n, resp., we set

$$\widetilde{\mathrm{occ}}(\tau, \sigma) := \binom{n}{k}^{-1} \cdot \# \left\{ \begin{array}{c} \text{occurrences of} \\ \tau \text{ in } \sigma \end{array} \right\} \in [0, 1].$$

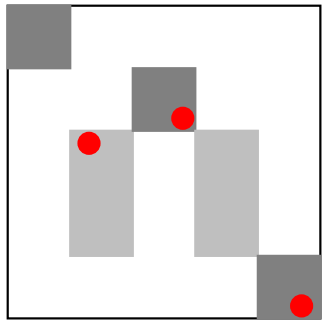
In other terms: take k elements uniformly at random in σ, the probability to find a pattern τ is $\widetilde{\mathrm{occ}}(\tau, \sigma)$.

Convergence and pattern density

This probabilistic interpretation extends to permutons: replacing σ with a permuton μ

$$\widetilde{\mathrm{occ}}(\tau, \mu) := \mathbb{P}^{\mu}(U^{(1)}, \cdots, U^{(k)} \text{ form a pattern } \tau),$$

where $U^{(1)}, \cdots, U^{(k)}$ are i.i.d. points in $[0,1]^2$ with distribution μ.



a "231 pattern" in a permuton

Theorem (Hoppen, Kohayakawa, Moreira, Rath, Sampaio, '13)

For each $n \geq 1$, let σ_n be a permutation of size n. TFAE

(a) *μ_{σ_n} converges to some permuton μ.*

(b) *For every pattern π, the proportion $\widetilde{\mathrm{occ}}(\pi, \sigma_n)$ tends to some δ_π*

Moreover, if both hold, $\delta_\pi = \widetilde{\mathrm{occ}}(\pi, \mu)$.

A detour through statistics

Problem statement: let (X, Y) be a bivariate r.v. such that the laws of X and Y have no atoms.

- Given a sample $(X_1, Y_1), \ldots, (X_n, Y_n)$, we want to test whether X and Y are independent;
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Simplification: we can apply increasing functions to X and Y, and assume that X and Y are uniform, i.e. that the law of (X, Y) is a permuton μ_0 (called copula by statistician).

$$X \text{ and } Y \text{ independent} \Leftrightarrow \mu_0 = \lambda_2,$$

where λ_2 is the uniform measure on $[0, 1]^2$.

Finding a test statistics

First proposition (Kendall, 1938): look at "inversions"

$$\#\{i < j : (X_i - X_j)(Y_i - Y_j) < 0\}.$$

Amounts to look at $d(21, \mu_0)$.

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On the other hand, λ_2 (as any permuton) is characterized by the values of all pattern densities $d(\tau, \mu)$.

Question: Something inbetween? A characterization of λ_2 by finitely many pattern densities?

Size 4 patterns are enough!

Theorem (Kral'–Pikhurko, '13)

Let μ be a permuton, and assume that, for all τ in S_4, we have $d(\tau, \mu) = d(\tau, \lambda_2)$. Then $\mu = \lambda_2$.

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Equivalently (permutation version):
if σ_n satisfies $d(\tau, \sigma_n) \to 1/24$ for all τ in S_4, then $d(\tau, \sigma_n) \to 1/k!$ for all $k \geq 1$ and all τ in S_k. (σ_n is then called quasi-random.)

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- General proof by Kral'–Pikhurko (arXiv 2012, journal 2013);
- Later improved (replacing S_4 by smaller sets or linear combination, and finding lower bounds on the size of such sets) in a series of papers (Bergsma, Chan, Crudele, Dassios, Drton, Dukes, Han, Král', Kurečka, Lee, Noel, Pehova, Shi, Sharifzadeh, Volec, 2020–2026).

Next few slides

According to time and taste:

- Sketch of proof of the S_5 result;
- Why is the S_3 not enough?

Inequalities and pattern densities

Let μ be a permuton and (X_1, Y_1), (X_2, Y_2), (X_3, Y_3) i.i.d. with law μ. Many natural quantities can be expressed through pattern densities: for example

$$\mathbb{P}[X_1 < X_2 < X_3 \wedge Y_2 < Y_1 < Y_3] = \frac{1}{6} d_{213}(\mu);$$

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Denote F_μ the cumulative distribution function of μ. The left-hand side writes

$$\mathbb{P}\big[\max(X_1, X_2) < X_3, Y_2 < Y_3\big] = \int_{[0,1]^2} x F_\mu(x, y) dF_\mu(x, y).$$

More generally, this kind of integral writes as linear combination of pattern densities.

Charaterizing λ_2 by size 5 patterns

For a permuton μ, define

$$\Delta(\mu) = \int_{[0,1]^2} (F_\mu(x,y) - xy)^2 dF_\mu(x,y).$$

Then $\Delta(\mu)$ is a linear combination of size 5 pattern densities.

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Assume $d_\tau(\mu) = d_\tau(\lambda_2)$ for all patterns τ of size 5. Then

$$\Delta(\mu) = \Delta(\lambda_2) = 0.$$

Hence $F_\mu(x,y) = xy$ for μ-almost all (x,y). This implies $\mu = \lambda_2$ (needs a bit of work). □

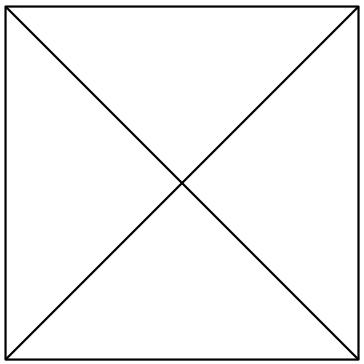
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Goal: find $\mu \neq \lambda_2$ with $d_\tau(\mu) = d_\tau(\lambda_2)$ for all τ of size 3.

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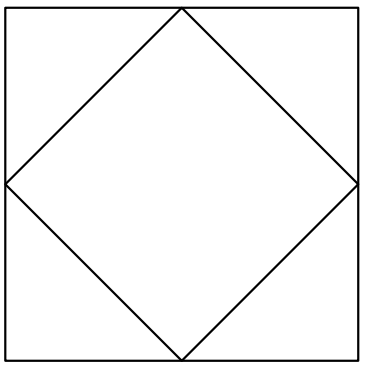
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μ_0



τ	d_τ
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321	$1/4$
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μ_1

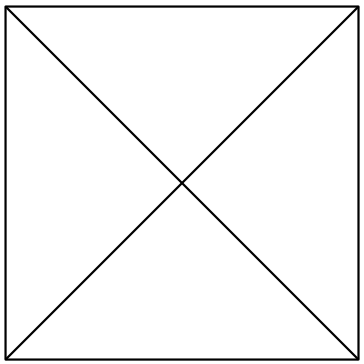


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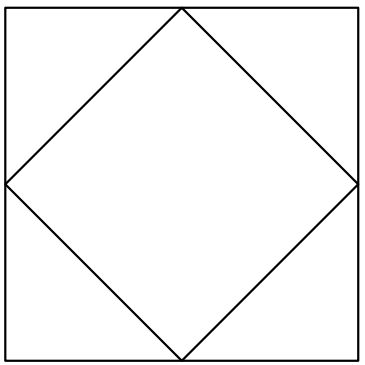
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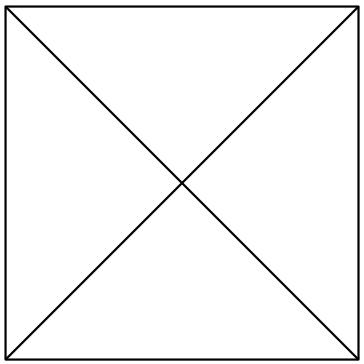
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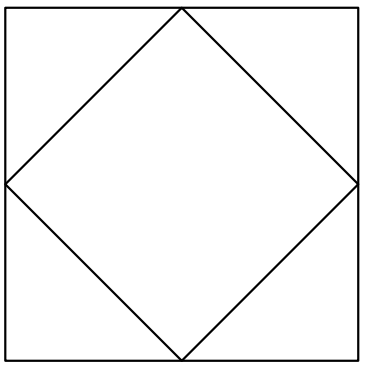
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- Using symmetry, if $d_{123}(\alpha\mu_0 + (1-\alpha)\mu_1) = 1/6$, then the same holds for all size 3 patterns;
- Intermediate value theorem: $d_{123}(\alpha\mu_0 + (1-\alpha)\mu_1) = 1/6$ for some α in $(0, 1)$. □