Sommes de nombres de Catalan, méandres et polynômes en 1/Pi

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Les midis de la Combinatoire
Nancy, 26 novembre 2024

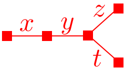
Problem statement

For a tree T, we consider

$$S(T) := \sum_{(x_e)\in\mathbb{Z}_+^{E(T)}} \left(\prod_{v\in V(T)} \mathrm{Cat}_{X_v}\, 4^{-X_v} \right),$$

where $\mathrm{Cat}_k = \frac{1}{k+1}\binom{2k}{k}$ and $X_v = \sum_{e\ni v} x_e$.

Example

Let $T =$ 

, then

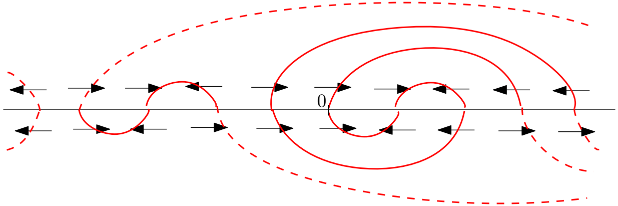
$$S(T) = \sum_{x,y,z,t\geq 0} \mathrm{Cat}_x\, \mathrm{Cat}_{x+y}\, \mathrm{Cat}_{y+z+t}\, \mathrm{Cat}_z\, \mathrm{Cat}_t\, 16^{-x-y-z-t}.$$

Our goal: compute these sums, and prove that they are in $\mathbb{Q}[\frac{1}{\pi}]$.

Motivation: meandric systems (1/5)

Definition (The Uniform Infinite Meandric System, or Infinite Noodle)

Draw two bi-infinite sequences of i.i.d. left/right arrows and close them in the unique non-crossing way.



Question

Is there an infinite connected component? What is the distribution of the size of the component of 0? In other words, compute $\mathbb{P}(|C_0| = k)$.

Motivation: meandric system (2/5)

Conjectures

- Almost surely, there is no infinite component (Curien–Kozma–Sidoravicius–Tournier, '19).
- As k tends to $+\infty$, we have $\mathbb{P}(|C_0| \geq k) \sim k^{-(2\sqrt{2}-1)/7+o(1)}$. (Borga–Gwynne–Park, '23).

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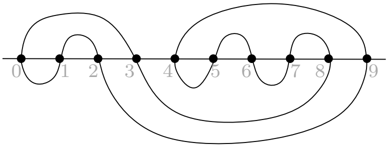
A related conjecture: define a meander as a **connected finite** arc configuration



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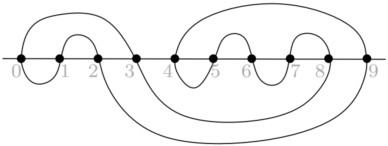
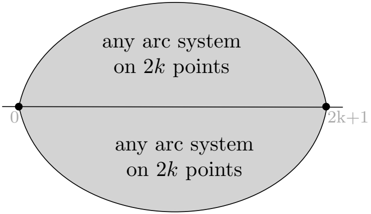


Conjecture (Di Francesco–Golinelli–Guitter, '00):

$$\#\{\text{meanders of size } 2n\} \sim C\, A^n n^{-\alpha},$$

with $\alpha = \dfrac{29+\sqrt{145}}{12}$.

A configuration with $|C_0| = 2$ looks like



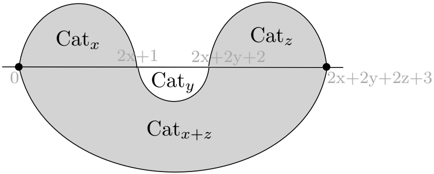
.

Hence

$$\mathbb{P}[|C_0| = 2] = 2 \sum_{k \geq 1} \mathrm{Cat}_k^2 \, 2^{-4k-4} = \tfrac{1}{8} \, S\big($$  $$\big).$$

Motivation: meandric system (4/5)

A configuration with $|C_0| = 4$ looks like



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Hence

$$\mathbb{P}[|C_0| = 4] = 4 \times 2 \times \sum_{x,y,z \geq 0} \mathrm{Cat}_x \, \mathrm{Cat}_y \, \mathrm{Cat}_z \, \mathrm{Cat}_{x+z} \, 2^{-4x-2y-4z-8}$$

$$= \frac{1}{32} S(\ \text{[image]}\ ) \, S(\ \text{[image]}\ ).$$

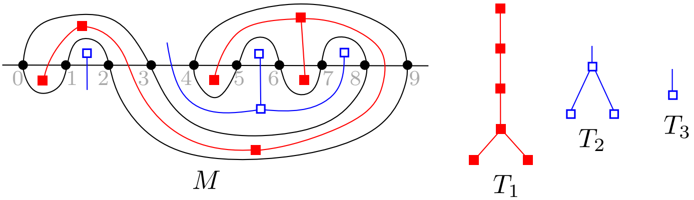
Motivation: meandric system (5/5)

More generally, to compute $\mathbb{P}[|C_0| = k]$,

- we sum over all possible component "shapes", i.e. over meanders of size k;
- for a meander M,

$$\mathbb{P}(S_0 = M) = 2^{-4k+1} k \prod_{i=1}^{d} S(T_i),$$

where the T_i's are the "dual trees" of the meander.



Quelques exemples en mathematica

Link to the notebook

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Results (P_n, P_n^h S_n are respectively the path, the path with a half-edge and the star with n vertices):

$$S(P_2) = \frac{16}{\pi} - 4;$$
$$S(P_2^h) = \frac{8}{\pi};$$
$$S(P_3) = 8 - \frac{64}{3\pi};$$
$$S(S_4) = \frac{64}{15\pi};$$
$$S(P_4) = -32 + \frac{64}{\pi} + \frac{128}{\pi^2}.$$

$S(P_2)$ and hypergeometric functions

We want to compute $S(P_2) = \sum_{x \in \mathbb{Z}_+} u_x$, where $u_x = \left(Cat_x 4^{-x}\right)^2$.

Note: the quotient u_{x+1}/u_x is a rational function in x. Such sums are called hypergeometric. Typical hypergeometric sums are

$$ {}_2F_1(a,b;c;z) := \sum_{n \geq 0} \frac{a^{\uparrow n} b^{\uparrow n}}{c^{\uparrow n}} \frac{z^n}{n!}, $$

where $u^{\uparrow n} := u(u+1)\cdots(u+n-1)$.

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In fact, we have

$$ S(P_2) = 4 \cdot {}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; 1\right) - 4. $$

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Lemma (Gauss identity)

If $c - a - b > 0$, we have

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

Thus ${}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1\right) = \frac{4}{\pi}$ and $S(P_2) = \frac{16}{\pi} - 4$.

$S(P_3)$ and the quadratic Catalan recurrence

We want to compute $S(P_3) = \sum_{x,y \in \mathbb{Z}_+} \mathrm{Cat}_x \, \mathrm{Cat}_y \, \mathrm{Cat}_{x+y} \, 16^{-x-y}$.

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Rewrite the sum using $Z = x + y$.

$$S(P_3) = \sum_{Z \geq 0} \mathrm{Cat}_Z \, 16^{-Z} \left(\sum_{\substack{x,y \geq 0 \\ x+y=Z}} \mathrm{Cat}_x \, \mathrm{Cat}_y \right)$$
$$= \sum_{Z \geq 0} \mathrm{Cat}_Z \, \mathrm{Cat}_{Z+1} \, 16^{-Z}.$$

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Again, this can be related to hypergeometric functions, namely

$$S(P_3) = 8 - 8 \cdot {}_2F_1\left(-\tfrac{1}{2}, \tfrac{1}{2}; 2; 1\right)$$

and Gauss identity allows to compute

$$S(P_3) = 8 - \frac{64}{3\pi}.$$

$S(P_2^h)$ and manipulating inequalities

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By symmetry, we also have

$$S(P_2^h) = \sum_{x\geq z\geq 0} \mathrm{Cat}_x \, \mathrm{Cat}_z \, 4^{-x-z}.$$

and thus

$$\begin{aligned} 2S(P_2^h) &= \sum_{x,z\geq 0} \mathrm{Cat}_x \, \mathrm{Cat}_z \, 4^{-x-z} + \sum_{\substack{x,z\geq 0 \\ x=z}} \mathrm{Cat}_x \, \mathrm{Cat}_z \, 4^{-x-z} \\ &= \left(\sum_{x\geq 0} \mathrm{Cat}_x \, 4^{-x} \right)^2 + S(P_2) = 4 + \left(\frac{16}{\pi} - 4 \right) = \frac{16}{\pi}. \end{aligned}$$

From edge-variables to vertex variables

We root T and write $v' \leq_T v$ if v' is a descendent of v. Color white (resp. black) vertices at even (resp. odd) distance from the root.

Lemma

The map sending $(x_e)_{e \in E(T)}$ to $(X_v)_{v \in V(T)}$ where $X_v = \sum_{e \ni v} x_e$, is injective

$$(*) \begin{cases} \sum_{w \in V_\circ(T)} X_w = \sum_{b \in V_\bullet(T)} X_b \\ \text{for all } v \in V_\circ(T),\ \sum_{w \in V_\circ(T), w \leq_T v} X_w \geq \sum_{b \in V_\bullet(T), b \leq_T v} X_b \\ \text{for all } v \in V_\bullet(T),\ \sum_{w \in V_\circ(T), w \leq_T v} X_w \leq \sum_{b \in V_\bullet(T), b \leq_T v} X_b. \end{cases}$$

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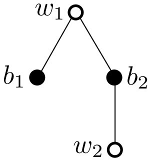
Corollary

$$S(T) = \sum_{\substack{(X_v) \in \mathbb{Z}_+^{V(T)} \\ (*)}} \prod_{v \in V} \mathrm{Cat}_{X_V} 4^{-V}.$$

Next example

A rooted version of P_4 is 

. We get

$$S(P_4) = \sum_{\substack{w_1,w_2,b_1,b_2 \geq 0 \\ w_1+w_2=b_1+b_2,\ b_2 \geq w_2}} \mathrm{Cat}_{w_1}\,\mathrm{Cat}_{w_2}\,\mathrm{Cat}_{b_1}\,\mathrm{Cat}_{b_2}\,4^{-w_1-w_2-b_1-b_2}.$$

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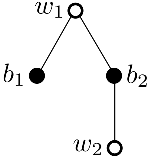
A rooted version of P_4 is 

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Manipulating the inequalities,

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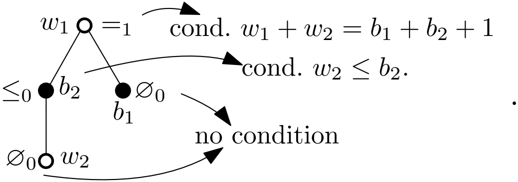
The first term can be computed using the quadratic recurrence, and the second is $S(P_2)^2$.

The general framework

- We consider rooted trees $\widetilde{T}$ with black/white/gray* vertices and vertex decorations in $\{=,\le,\ge,\varnothing\}\times\mathbb{Z}$.
- To each vertex is associated a condition (C_v) comparing the sums of white variables and black variables below it.
- We consider the sum

$$S(\widetilde{T}) := \sum_{(X_v)\in\mathbb{Z}_+^{V_{\bullet/\circ}(T)}} \left(\prod_{v\in V_{\bullet/\circ}(T)} \mathrm{Cat}_{X_v}\, 4^{-X_v} \right) \left(\prod_v 1[C_v] \right).$$

Example: A decorated version of P_4



* Gray vertices do not have associated variables but may impose conditions.

Our main result (general version)

Theorem (Bostan, F., Thévenin, '25)

For any colored decorated tree $S(\widetilde{T})$, the sum $S(\widetilde{T})$ belongs to $\mathbb{Q}[\frac{1}{\pi}]$.

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For any fixed k, $\mathbb{P}[|C_0| = k]$ is in $\mathbb{Q}[\frac{1}{\pi}]$.

More generally, we can replace 4^{-X_v} by t^{X_v} and we have

Theorem (Bostan, F., Thévenin, '25)

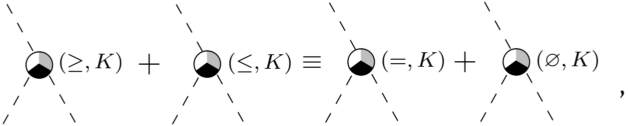
The series $S(\widetilde{T})(t)$ belongs to the space

$$\mathbb{Q}\Big[t^{\pm 1}, \sqrt{1-4t}, {}_2F_1\big(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; 16t^2\big), {}_2F_1\big(-\tfrac{1}{2}, \tfrac{1}{2}; 2; 16t^2\big)\Big].$$

Moreover, ${}_2F_1\big(-\frac{1}{2}, -\frac{1}{2}; 1; 16t^2\big)$ and ${}_2F_1\big(-\frac{1}{2}, \frac{1}{2}; 2; 16t^2\big)$ are algebraically independent over $\mathbb{Q}\big[t^{\pm 1}, \sqrt{1-4t}\big]$.

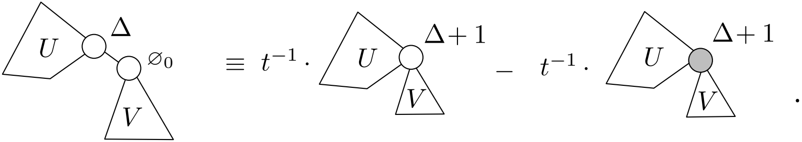
Some words on the proof

We have some reduction rules, like



(manipulating inequalities)

or



(using the quadratic recurrence)

Reducing any tree with such rules?

Let T be a bicolored tree. We start from the leaves.

- If there are two twin leaves of the same color or if one leaf has the same color than its parents, we can apply the quadratic recurrence 🙂

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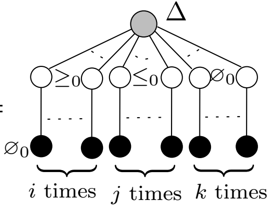
- The difficult case is that of long stars, e.g.

$V^{\Delta}_{i,j,k} :=$ 

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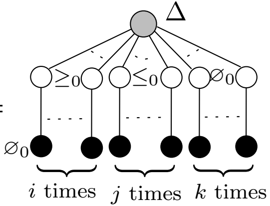
$V^{\Delta}_{i,j,k} :=$ 

.

By *reversing inequalities*, we obtain relations between the $V^{\Delta}_{i,j,k}$ for various values of i, j and k.

A linear system for long stars

Lemma

For any "rootstock" R, any decoration Δ and any $d \geq 2$, we have

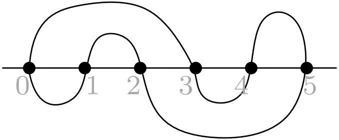
$$\begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 2 & 1 \\ 0 & \dots & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} S(R \mid V^{\Delta}_{1,d-1,0}) \\ \\ \vdots \\ \\ S(R \mid V^{\Delta}_{d-1,1,0}) \end{pmatrix} = \begin{pmatrix} X_1 \\ \\ \vdots \\ \\ X_{d-1} \end{pmatrix} - \begin{pmatrix} S(R|V^{\Delta}_{0,d,0}) \\ 0 \\ \vdots \\ 0 \\ S(R|V^{\Delta}_{d,0,0}) \end{pmatrix},$$

where, for $1 \leq i \leq d-1$,

$$X_i = S(R \mid V^{\Delta}_{i-1,d-1-i,2}) + 2S(R \mid V^{\Delta}_{i-1,d-1-i,1}) \cdot S_{=,0} + S(R \mid V^{\Delta}_{i-1,d-1-i,0}) \cdot S^2_{=,0}.$$

→ this lemma allows to express $S(R \mid V^{\Delta}_{i,j,0})$ in terms of smaller trees since the matrix is invertible!

Thanks for your attention!



$$\sum_{x,y,z} \mathrm{Cat}_x \, \mathrm{Cat}_{x+y} \, \mathrm{Cat}_{y+z} \, \mathrm{Cat}_z \, 16^{-x-y-z} = -32 + \frac{64}{\pi} + \frac{128}{\pi^2}.$$

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