Convergence laws for random permutations

Valentin Féray
(joint work with Michael Albert, Mathilde Bouvel and Marc Noy)

CNRS, Institut Élie Cartan de Lorraine

Dagstuhl seminar on Logic and Random Structures,
February 2022
General problem

Definition

A sequence of random permutations $\sigma_n \in S_n$ satisfies:

- a **0-1 law** if, for every first-order property $\Psi$, the probability $\mathbb{P}[\sigma_n \models \Psi]$ tends to 0 or 1;
- a **convergence law** if, for every first-order property $\Psi$, the probability $\mathbb{P}[\sigma_n \models \Psi]$ has a limit as $n \to \infty$?

Motivation: large literature on 0-1/convergence law for random graphs on one side, on random permutations on the other sides.

Goal of today’s talk: panorama of some results/questions on the topic.
Permutations as models of a logical theory

Two ways to see permutations

Bijection point of view

A permutation \( \sigma \) is a pair

\((A, f)\),

where \( f \) a bijection from \( A \to A \).

Matrix point of view

A permutation \( \sigma \) is a triple

\((A, <_P, <_V)\),

where \( <_P \) and \( <_V \) are linear orders on \( A \).

\[
\begin{align*}
1 & \rightarrow 3; \quad f(2) = 5; \quad f(3) = 4; \\
2 & \rightarrow 5; \quad f(4) = 1; \quad f(5) = 2.
\end{align*}
\]

\[
\begin{array}{cccc}
& D & & \\
A & C & & \\
& B & & E
\end{array}
\]

\[
A <_P B <_P C <_P D <_P E; \\
B <_V E <_V A <_V C <_V D.
\]
Permutations as models of a logical theory

Two ways to see permutations

**Bijection point of view**

A permutation $\sigma$ is a pair

$$(A, f),$$

where $f$ a bijection from $A \rightarrow A$.

$\sigma = (\{1, 2, 3, 4, 5\}, f)$,

$f(1) = 3; f(2) = 5; f(3) = 4; f(4) = 1; f(5) = 2$.

TOOB: “theory of one bijection”

**Matrix point of view**

A permutation $\sigma$ is a triple

$$(A, <_P, <_V),$$

where $<_P$ and $<_V$ are linear orders on $A$.

$\sigma = (\{1, 2, 3, 4, 5\}, <_P, <_V)$,

$A <_P B <_P C <_P D <_P E$;

$B <_V E <_V A <_V C <_V D$.

TOTO: “theory of two orders”
First order properties

**Definition:** A first order formula is written using variables $x, y, z, \ldots$, relational symbols $f$ or $<_P, <_V$, and logical symbols $\exists, \forall, =, \neg$ (we quantify only on variables, not on sets of variables).
First order properties

Definition: A first order formula is written using variables $x, y, z, \ldots$, relational symbols $f$ or $<_P, <_V$, and logical symbols $\exists, \forall, =, \neg$ (we quantify only on variables, not on sets of variables).

Bijection point of view (TOOB)

Example: existence of a fixed point

$$\exists x : f(x) = x$$

More generally, one can express properties regarding short cycles of the permutation (conjugate permutations are isomorphic!)
First order properties

**Definition:** A first order formula is written using variables $x, y, z, \ldots$, relational symbols $f$ or $<_P, <_V$, and logical symbols $\exists, \forall, =, \neg$ (we quantify only on variables, not on sets of variables).

---

**Bijection point of view (TOOB)**

**Example:** existence of a fixed point

$$\exists x : f(x) = x$$

More generally, one can express properties regarding short cycles of the permutation (conjugate permutations are isomorphic!)

**Matrix point of view (TOTO)**

**Example:** existence of a 213 pattern

$$\exists x, y, z : (x <_P y <_P z) \land (y <_V x <_P z)$$

More generally, one can express many properties regarding “generalized pattern”... but not the existence of a fixed point!
An “incompatibility” result

Recall that the support of a permutation is the set of its non-fixed points.

**Theorem (Albert, Bouvel, F., ’20)**

Let \((P)\) be a property, expressible as a first-order formula for both TOOB and TOTO. Then

- either all permutations with sufficiently large support verify \((P)\),
- or there is a bound on the size of the support of permutations verifying \((P)\).

Proof uses Ehrenfeucht–Fraïssé games and combinatorial arguments.
Proposition (folklore?)

Let $\sigma_n$ a uniform random permutation of $n$. Then $\sigma_n$ admits a convergence law for TOOB.

"Proof:" it is well-known that

- $\sigma_n$ contains a large cycle with high probability;
- the small cycle counts ($#C_1(\sigma), #C_2(\sigma), \ldots$) converge jointly to Poisson random variables.
Proposition (folklore?)

Let $\sigma_n$ a uniform random permutation of $n$. Then $\sigma_n$ admits a convergence law for TOOB.

“Proof:” it is well-known that

- $\sigma_n$ contains a large cycle with high probability;
- the small cycle counts ($\#C_1(\sigma), \#C_2(\sigma), \ldots$) converge jointly to Poisson random variables.

Theorem (Compton, '87)

Permutations admit an unlabelled 0-1 law. Namely, if $\sigma_n$ is in the conjugacy class $C_\lambda$, where $\lambda$ is taken uniformly at random among partitions of $n$, then $\sigma_n$ admits a 0-1 law for TOOB.

(This is one application of a general result, relating 0-1 law and analytic combinatorics.)
Theorem (Foy, Woods, '90)

Let $\sigma_n$ be a uniform random permutation. Then $\sigma_n$ does not admit a convergence law for TOTO (matrix point of view).

In fact, they prove that there exists a first-order property $\Psi$ (using the relations $<_V$, $<_P$) such that

$$\liminf P(\sigma_n \models \Psi) = 0, \quad \limsup P(\sigma_n \models \Psi) = 1.$$
Random pattern avoiding permutation

Definition

An occurrence of a pattern \( \tau \) in \( \sigma \) is a subsequence \( \sigma_{i_1} \ldots \sigma_{i_k} \) that is order-isomorphic to \( \tau \), i.e. \( \sigma_{i_s} < \sigma_{i_t} \iff \tau_s < \tau_t \).

Example (occurrences of 213)

\[
\begin{align*}
2 & 4 & 5 & 3 & 6 & 1 \\
8 & 2 & 3 & 4 & 6 & 1 & 7 & 5
\end{align*}
\]

We denote \( \text{Av}_n(\tau_1, \ldots, \tau_r) \) the set of permutations \( \sigma \) of size \( n \) avoiding \( \tau_1, \ldots, \tau_r \). For fixed \( \tau_1, \ldots, \tau_r \), we consider a uniform random permutation \( \sigma_n \) in \( \text{Av}_n(\tau) \).
Theorem (Albert, Bouvel, F., Noy ’22)

For each \( n \geq 1 \), let \( \sigma_n \) be a uniform random permutation in \( \text{Av}_n(231) \). Then \( \sigma_n \) satisfies a convergence law.

The proof uses analytic combinatorics; it is based on Woods’ approach for convergence law for rooted trees (’97).

Theorem (Braunfeld, Kukla ’22)

For each \( n \geq 1 \), let \( \sigma_n \) be a uniform random permutation in \( \text{Av}_n(231, 312) \) (layered permutations). Then \( \sigma_n \) satisfies a convergence law.