Formes limites de permutations aléatoires

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Main topic: random permutations

- Classical questions: look at some statistics, like the number of cycles (of given length), longest increasing subsequences, ... (usually for uniform or Ewens distributions)

- a more recent approach: look for a limit theorem for the renormalized "permutation matrix" (interesting for non-uniform or constrained models).
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- Classical questions: look at some statistics, like the number of cycles (of given length), longest increasing subsequences, ... (usually for uniform or Ewens distributions)

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Here: we consider some constraints, called pattern avoidance.
A permutation $\pi$ can be represented by its diagram (\sim permutation matrix) and mapped to a probability measure $\mu_\pi$ on $[0,1]^2$, called\textit{ permuton}.

$$\pi = 52413 = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \\
\end{array} \qquad \mapsto \qquad \mu_\pi = \begin{array}{ccc}
\begin{array}{c}
\text{gray squares}
\end{array}
\end{array}$$

In $\mu_\pi$, each small square has weight $1/n$ (i.e. density $n$).

We have a natural notion of limit for such objects: the \textit{weak convergence} of measure.
Permutation patterns

Definition
An occurrence of a pattern $\tau$ in $\sigma$ is a subsequence $\sigma_{i_1} \ldots \sigma_{i_k}$ that is order-isomorphic to $\tau$, i.e. $\sigma_{i_s} < \sigma_{i_t} \iff \tau_s < \tau_t$.

Example (occurrences of 213)

245361
82346175

Pattern avoidance is a well-studied concept in enumerative combinatorics!
Uniform random permutations avoiding some patterns

no constraints

Av(231) (©MM)

Av(321) (©HRS)

Av(4231) (©NM)

Av(2413,3142) (©MM)

Av(… ) (©MM)

Detour: an operad structure on permutations

Well-known: the set $S_n$ of permutations of size $n$ is a group for the composition operation.

Less known: the set $\bigcup_{n \geq 1} S_n$ of permutations of all sizes is an operad for the substitution operation.
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Definition (substitution)

Let $\theta$ be a permutation of size $d$ and $\pi^{(1)}, \ldots, \pi^{(d)}$ be permutations. The diagram of the permutation $\theta[\pi^{(1)}, \ldots, \pi^{(d)}]$ is obtained by replacing the $i$-th dot in the diagram of $\theta$ with the diagram of $\pi^{(i)}$ (for each $i$).

$$2413[132, 21, 1, 12] = 12 = 24387156$$
Simple permutations and substitution decomposition

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A permutation is called *simple* if it cannot be obtained as a nontrivial substitution.

Examples: 12, 21, 3142, 2413, 25314, …
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It’s an analogue notion to that of prime numbers. In both cases, there are "factorization theorems":

- an integer can be uniquely represented as a multiset of prime numbers;
- a permutation can be (almost) uniquely represented as a "tree of permutations" (we call this its substitution decomposition)

We get trees and not multisets since we have an operad structure, and not a commutative monoid (as for integers).
Substitution decomposition and separable permutations

Inner nodes of the decomposition tree are labelled with simple permutations.

Proposition $\mathcal{A}_v(2413, 3142)$ is the set of permutations whose decomposition trees contain only nodes labelled with $12$ and $21$. These are called separable permutations.
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**Proposition**

\( \text{Av}(2413, 3142) \) is the set of permutations whose decomposition trees contain only nodes labelled with 12 and 21.

These are called **separable** permutations.
Problem

Given the tree $T$ associated with a separable permutation $\sigma$ and integers $i < j$, how to determine whether $\sigma(i) < \sigma(j)$?

Answer: look at the decoration of the first common ancestor between the $i$-th leaf and the $j$-th leaf. In the example, it is 21 so $\sigma(2) > \sigma(5)$.

Write $i < T j$ (resp. $i > T j$) when $i < j$ and their common ancestor is 12 (resp. 21). We can reconstruct $\sigma$ from this order: $\sigma(i) = 1 + |\{j : j < T i\}|$.
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$$\sigma(i) = 1 + \left| \{j : j <_T i \} \right|$$
The limiting object: the Brownian separable permuton

\[ e \] is a Brownian excursion and \( S : \text{LocalMin}(e) \to \{\oplus, \ominus\} \) is an assignment of i.i.d. random signs to local minima of \( e \) (the probability to get \( \oplus \) is \( p \in (0,1) \)).

(the Brownian excursion encodes the limit of the trees, its local minima corresponding to branching points in the trees)
If \( x < y \) in \([0,1]\), we set \( x <_{(e,S)} y \) if \( S(\arg\min_{[x,y]} e) = \oplus \)
and \( y <_{(e,S)} x \) if \( S(\arg\min_{[x,y]} e) = \ominus \).

2. We define a function \( \tau : [0,1] \to [0,1] \) by \( \tau(x) = \text{Leb}(\{y : y <_{(e,S)} x\}) \).

3. The **Brownian separable permuton** \( \mu_p \) is the corresponding permuton.
Limits of separable permutations

Theorem (Bassino-Bouvel-F.-Gerin-Pierrot, 2018)

For each $n$, let $\sigma_n$ be a uniform random separable permutation of size $n$. Then $\mu_{\sigma_n}$ converges in distribution to $\mu_{1/2}$.
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Similar results for other classes $\text{Av}(B)$ (based on the algebraic properties of the class w.r.t. the operad structure):

- uniform random permutations $\sigma_n$ in substitution-closed classes $\text{Av}(B)$ tend to $\mu_p$ (under some analytic conditions; $p$ depends on the class).
- If $\text{Av}(B)$ is finitely generated (i.e. contains finitely many simple), there is a dichotomy (see next slide; again with some technical conditions).
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The dichotomy for finitely generated classes

"Essentially branching case"

"Essentially linear case"
Any questions? I have one... 

Let \( \sigma \) be a permutation of size \( n \). Do there exist polynomials \( P_1, P_2, \ldots, P_n \) such that

- \( P_1(0) = \cdots = P_n(0) = 0 \);
- for small \( x < 0 \), we have \( P_1(x) < P_2(x) < \cdots < P_n(x) \);
- for small \( x > 0 \), we have \( P_{\sigma(1)}(x) < P_{\sigma(2)}(x) < \cdots < P_{\sigma(n)}(x) \)?
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Answers in office 221 (Nancy). Thank you for your attention.