Dual combinatorics of Jack polynomials

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What is this talk about?

- **Symmetric functions:**

\[ x_1^3 + x_2^3 + x_3^3 + \ldots \]

\[ \sum_{i < j} x_i x_j \]
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- Symmetric functions.
- in particular Jack polynomials $J^{(\alpha)}_{\lambda}$.

\[
J^{(\alpha)}_{(2)} = (\alpha + 1) \cdot x_1^2 + 2 \cdot x_1 \cdot x_2 + (\alpha + 1) \cdot x_2^2 \\
+ 2 \cdot x_1 \cdot x_3 + 2 \cdot x_2 \cdot x_3 + (\alpha + 1) \cdot x_3^2 + \ldots
\]
What is this talk about?

- Symmetric functions.
- in particular **Jack polynomials** $J^{(\alpha)}_\lambda$.
- We present a new approach to the study of Jack polynomials (called **dual**), due to Michel Lassalle with a lot of open questions.
What is this talk about?

- Symmetric functions.
- in particular Jack polynomials $J^{(\alpha)}_{\lambda}$.
- We present a new approach to the study of Jack polynomials (called dual), due to Michel Lassalle with a lot of open questions.
- Partial answers (for $\alpha = 1, 2$) involve combinatorics and representation theory.
Outline of the talk

1. Definitions and notations
2. Dual approach and Lassalle’s conjectures
3. Solution to the $\alpha = 1$ case using Young symmetrizer
4. Overview of the $\alpha = 2$ case
5. Leads towards the general case
Partitions

Definition

A partition (of $n$) is a non-increasing list of integer (of sum $n$). If $\lambda$ is a partition of $n$, we denote $\lambda \vdash n$.

Example: $(4, 3, 1) \vdash 8$.

Graphical representation as Young diagram:
Symmetric functions

Definition
A symmetric function is a symmetric polynomial in infinitely many variables $x_1, x_2, \ldots$.

\begin{itemize}
  \item bounded degree ;
  \item when we set $x_{n+1} = x_{n+2} = \cdots = 0$, we have a symmetric polynomial in $x_1, \ldots, x_n$.
\end{itemize}

Examples:

\begin{align*}
  p_3 &= x_1^3 + x_2^3 + x_3^3 + \ldots, \\
  e_2 &= \sum_{i<j} x_i x_j \\
\end{align*}

Swaping the indices of two variables does not change the polynomials.
Symmetric functions

Definition
A symmetric function is a symmetric polynomial in infinitely many variables $x_1, x_2, \ldots$.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition. Set

$$m_\lambda(x_1, x_2, \ldots) = x_1^{\lambda_1} \ldots x_r^{\lambda_r} + \text{its images by swaping indices.}$$

Proposition
The family $(m_\lambda)_\lambda$ partition is a linear basis of the symmetric function ring.

called monomial basis.
Symmetric functions

Definition
A symmetric function is a symmetric polynomial in infinitely many variables $x_1, x_2, \ldots$.

Set $p_0 = 1$, $p_k = x_1^k + x_2^k + \ldots$. Power sums

Proposition
The family $(p_i)_{i \geq 1}$ is an algebraic basis of the symmetric function ring. In other words, any symmetric function writes uniquely as a linear function of

$$
\left( p_\lambda = \prod_{i} p_{\lambda_i} \right),
$$

where $\lambda$ runs over partitions.
Definitions and notations

Schur functions

Definition (Jacobi, 1841)
Let $\lambda$ be a partition. Define

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det \left( x_i^{\lambda_j + n-j} \right)}{\det \left( x_i^{n-j} \right)}.$$ 

Then $(s_\lambda)$ is a linear basis of symmetric function ring.

Example:

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 \cdot x_2 + x_1 \cdot x_2^2 + x_1^2 \cdot x_3 + 2 \cdot x_1 \cdot x_2 \cdot x_3$$
$$+ x_2^2 \cdot x_3 + x_1 \cdot x_3^2 + x_2 \cdot x_3^2$$
Representation theory of symmetric group

- $S_n$: group of permutations of $n$.

- We are interested in its representation that is group morphisms $S_n \rightarrow \text{GL}(V)$, $V \in \mathbb{C}$-vector space of finite dimension.
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- What the general theory says us:
  - it is enough to study the irreducible representations.
  - these irreducible representations $\rho^\lambda$ are enumerated by the number of conjugacy classes in $S_n$, that is of partitions of $n$. 
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- We are interested in its representation that is group morphisms $S_n \rightarrow GL(V)$, $V \subseteq \mathbb{C}$-vector space of finite dimension.

- what the general theory says us:
  - it is enough to study the irreducible representations.
  - these irreducible representations $\rho^\lambda$ are enumerated by the number of conjugacy classes in $S_n$, that is of partitions of $n$.
  - what is really important is to compute characters ($=\text{trace}$), that is a collections of numbers

\[ \chi^\lambda_\mu := \text{tr}(\rho^\lambda(\pi)) \quad \text{(with $\pi$ of cycle type $\mu$)} \]

indexed by two partitions.
Frobenius formula

**Theorem (Frobenius, 1900)**

Let $\lambda$ be a partition of $n$, then

$$s_\lambda = \sum_{\mu \vdash n} \chi_\mu^\lambda \frac{p_\mu}{z_\mu},$$

where $z_\mu = \prod_{i \geq 1} i^{m_i} m_i !$ if $\mu$ has $m_1$ parts equal to 1, . . .

This result makes a link between two different theories: symmetric functions and representation theory of the symmetric group.
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Hall scalar product is defined by $\langle p_\mu, p_\nu \rangle := z_\mu \delta_{\mu, \nu}.$
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Orthonormality of irreducible characters $\Rightarrow \langle s_\lambda, s_\rho \rangle = \delta_{\lambda,\rho}$.
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Theorem (Frobenius, 1900)
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Hall scalar product is defined by $\langle p_\mu, p_\nu \rangle := z_\mu \delta_{\mu, \nu}$.

Orthonormality of irreducible characters $\Rightarrow \langle s_\lambda, s_\rho \rangle = \delta_{\lambda, \rho}$

Proposition
The basis $(s_\lambda)$ may be obtained from the monomial basis by Gram-Schmidt orthonormalization process. (use lexicographic order on partitions).
Consider the following deformation of Hall scalar product:

\[ \langle p_{\mu}, p_{\nu} \rangle_{\alpha} = \alpha^{\ell(\mu)} z_{\mu} \delta_{\mu, \nu} \]

\(\ell(\mu)\): length (number of parts) of the partition \(\mu\).

**Definition**

**Jack polynomials** \(P_{\alpha}^{Q_{\lambda}}\) are obtained from the monomial basis by Gram-Schmidt orthonormalization process (with respect to the deformed scalar product).
Jack polynomials

Consider the following deformation of Hall scalar product:

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\( \ell(\mu) \): length (number of parts) of the partition \( \mu \).

Definition

Jack polynomials \( P_{\lambda}^{(\alpha)} \) are obtained from the monomial basis by Gram-Schmidt orthonormalization process (with respect to the deformed scalar product).

Renormalization: \( J_{\lambda}^{(\alpha)} = c_{\lambda}^{(\alpha)} P_{\lambda}^{(\alpha)} \) with \( c_{\lambda}^{(\alpha)} \) explicit.
Consider the following deformation of Hall scalar product:

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\langle p_\mu, p_\nu \rangle_\alpha = \alpha^{\ell(\mu)} z_\mu \delta_{\mu,\nu}
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\(\ell(\mu)\): length (number of parts) of the partition \(\mu\).

**Definition**

Jack polynomials \(PQ^{(\alpha)}_\lambda\) are obtained from the monomial basis by Gram-Schmidt orthonormalization process (with respect to the deformed scalar product).

Renormalization: \(J^{(\alpha)}_\lambda = c^{(\alpha)}_\lambda PQ^{(\alpha)}_\lambda\) with \(c^{(\alpha)}_\lambda\) explicit.

Specialization: \(J^{(1)}_\lambda = c^{(1)}_\lambda s_\lambda = \frac{n!}{\dim(V_\lambda)} s_\lambda\).

\(V_\lambda\): irreducible representation of \(S_n\) associated to \(\lambda\).
Jack “characters”

Main object in the talk

Let \( \lambda \) and \( \mu \) be partitions of \( n \). Define \( \theta_{\mu}^{\lambda, (\alpha)} \) by

\[
j_{\lambda}^{(\alpha)} = \sum_{\mu \vdash n} \theta_{\mu}^{\lambda, (\alpha)} \cdot p_{\mu}.
\]
Jack “characters”

Main object in the talk

Let $\lambda$ and $\mu$ be partitions of $n$. Define $\theta_\mu^{\lambda, (\alpha)}$ by

$$J^{(\alpha)}_{\lambda} = \sum_{\mu \vdash n} \theta_\mu^{\lambda, (\alpha)} \cdot p_\mu.$$ 

Unfortunately, $\theta_\mu^{\lambda, (\alpha)}$ has no (known) representation-theoretical interpretation for general $\alpha$. 

Jack “characters”

Main object in the talk

Let $\lambda$ and $\mu$ be partitions of $n$. Define $\theta_{\mu}^{\lambda,(\alpha)}$ by

$$J_{\lambda}^{(\alpha)} = \sum_{\mu \vdash n} \theta_{\mu}^{\lambda,(\alpha)} \cdot p_{\mu}.$$ 

Unfortunately, $\theta_{\mu}^{\lambda,(\alpha)}$ has no (known) representation-theoretical interpretation for general $\alpha$.

But, it shares (conjecturally) a lot of properties with

$$\theta_{\mu}^{\lambda,(1)} = z_{\mu} n! \frac{\chi_{\mu}^{\lambda}}{\dim(\lambda)},$$

whence the name **Jack characters**.
A function on the set of all Young diagrams

Definition
Let $\mu$ be a partition of $k$ without part equal to 1. Define

$$
\text{Ch}_{\mu}^{(\alpha)}(\lambda) = \begin{cases} 
  z_{\mu} \theta_{\mu \lambda, (\alpha)}^{1_{n-k}} & \text{if } n = |\lambda| \geq k; \\
  0 & \text{otherwise.}
\end{cases}
$$

$\text{Ch}_{\mu}^{(\alpha)}$ is a function of all Young diagrams.
A function on the set of all Young diagrams

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Ch_{\mu}^{(\alpha)}(\lambda) = \begin{cases} 
   z_{\mu} \theta_{\mu 1}^{\lambda,(\alpha)} & \text{if } n = |\lambda| \geq k; \\
   0 & \text{otherwise.}
\end{cases}
\]

\( Ch_{\mu}^{(\alpha)} \) is a function of all Young diagrams.

**Specialization:** if \( |\mu| < |\lambda| \),

\[
Ch_{\mu}^{(1)}(\lambda) = \frac{|\lambda|!}{(|\lambda| - |\mu|)!} \cdot \frac{\chi_{\mu 1}^{\lambda 1} n-k}{\dim(V_\lambda)}.
\]

Introduced by S. Kerov, G. Olshanski in the case \( \alpha = 1 \), by M. Lassalle in the general case.
A function on the set of all Young diagrams

Definition
Let $\mu$ be a partition of $k$ without part equal to 1. Define

$$\text{Ch}^{(\alpha)}_{\mu}(\lambda) = \begin{cases} z_{\mu}^{\lambda, (\alpha)} \mu_{1}^{n-k} & \text{if } n = |\lambda| \geq k; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition (M. Lassalle)
For any $r$, the application

$$(\lambda_1, \ldots, \lambda_r) \mapsto \text{Ch}^{(\alpha)}_{\mu}((\lambda_1, \ldots, \lambda_r))$$

is a polynomial in $\lambda_1, \ldots, \lambda_r$. Besides, it is symmetric in $\lambda_1 - 1, \ldots, \lambda_r - r$.

In other words, $\text{Ch}^{(\alpha)}_{\mu}$ is a shifted symmetric function.
Consider two lists \( p \) and \( q \) of positive integers of the same size, with \( q \) non-decreasing. We associate to them the partition

\[
\lambda(p, q) = (q_1, \ldots, q_1, q_2, \ldots, q_2, \ldots).
\]

You can see the Young diagram of \( \lambda(p, q) \) below.
Multirectangular coordinates (R. Stanley)

Consider two lists $p$ and $q$ of positive integers of the same size, with $q$ non-decreasing.
We associate to them the partition

$$\lambda(p, q) = (\underbrace{q_1, \ldots, q_1}_{p_1 \text{ times}}, \underbrace{q_2, \ldots, q_2}_{p_2 \text{ times}}, \ldots).$$

Conjecture (M. Lassalle)

Let $\mu$ be a partition of $k$. $(-1)^k \operatorname{Ch}_{\mu}^{(\alpha)}(\lambda(p, q))$ is a polynomial in

$$p_1, p_2, \ldots, -q_1, -q_2, \ldots, \alpha - 1$$

with non-negative integer coefficients.

Polynomiality in $p$ and $q$: consequence of shifted symmetry
Polynomiality in $\alpha$: F., Dołęga 2012
Multirectangular coordinates (R. Stanley)

Consider two lists $p$ and $q$ of positive integers of the same size, with $q$ non-decreasing. We associate to them the partition

$$\lambda(p, q) = (q_1, \ldots, q_1, q_2, \ldots, q_2, \ldots).$$

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Let $\mu$ be a partition of $k$. $(-1)^k \text{Ch}_{\mu}^{(\alpha)}(\lambda(p, q))$ is a polynomial in

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with non-negative integer coefficients.

Hard, interesting still open part: non-negativity (and integrity).
Case $\alpha = 1$

Goal of the next few slides: sketch the proof of Lassalle’s conjecture in the case $\alpha = 1$.


Let $\mu$ be a partition of $k$. $(-1)^k \text{Ch}_{\mu}^{(1)}(\lambda(p, q))$ is a polynomial in $p_1, p_2, \ldots, -q_1, -q_2, \ldots$ with non-negative integer coefficients.

Reminder: if $|\mu| < |\lambda|,$

$$\text{Ch}_{\mu}^{(1)}(\lambda) = \frac{|\lambda|!}{(|\lambda| - |\mu|)!} \cdot \frac{\chi_{\mu 1n-k}^\lambda}{\dim(V_\lambda)}.$$  

Hence, we need to know how to compute $\chi_{\mu 1n-k}^\lambda$.  

Next step: construction of irreducible representations of $S_n$.  

Young’s symmetrizer (1/3)

Let $\lambda$ be a partition of $n$.

Choose a filling $T_0$ of $\lambda$.

Example:

$\lambda = (2, 2)$, $T_0 = \begin{array}{cc} 2 & 4 \\ 1 & 3 \end{array}$. 
Let $\lambda$ be a partition of $n$.

Choose a filling $T_0$ of $\lambda$. Define

$$a_\lambda = \sum_{\sigma \in S_n} \sigma \in \mathbb{C}[S_n],$$

where $\text{RS}(T_0)$ is the row stabilizer of $T_0$;

Example:

$$\lambda = (2, 2), \quad T_0 = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}.$$

$$a_\lambda = \text{id} + (1 \ 3) + (2 \ 4) + (1 \ 3)(2 \ 4)$$

Everything depends on $T_0$, although that is hidden in notations.
Young’s symmetrizer (1/3)

Let $\lambda$ be a partition of $n$.

Choose a filling $T_0$ of $\lambda$. Define

$$a_\lambda = \sum_{\sigma \in S_n, \sigma \in RS(T_0)} \sigma \in \mathbb{C}[S_n],$$

where $RS(T_0)$ is the row stabilizer of $T_0$;

$$b_\lambda = \sum_{\tau \in S_n, \tau \in CS(T_0), \in \mathbb{C}[S_n]} \varepsilon(\tau) \tau$$

$CS(T_0)$ is the column stabilizer of $T_0$.

Example:

$\lambda = (2, 2)$, $T_0 = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$.

$$a_\lambda = \text{id} + (1 \, 3) + (2 \, 4) + (1 \, 3)(2 \, 4)$$

$$b_\lambda = \text{id} - (1 \, 2) - (3 \, 4) + (1 \, 2)(3 \, 4)$$

Everything depends on $T_0$, although that is hidden in notations.
Consider
\[ a_\lambda \cdot b_\lambda = \sum_{\sigma \in S_n}^{\sigma \in \text{RS}(T_0)} \sum_{\tau \in S_n}^{\tau \in \text{CS}(T_0)} \varepsilon(\tau) \sigma \tau \]

**Lemma**

Then \( p_\lambda = \alpha_\lambda a_\lambda \cdot b_\lambda \) is a projector \((i.e. \ p_\lambda^2 = p_\lambda)\) for a well-chosen constant \( \alpha_\lambda \).
Reminder: $p_\lambda = \alpha_\lambda a_\lambda \cdot b_\lambda$ is a projector.
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Set $V_\lambda = \mathbb{C}[S_n]p_\lambda$, subspace of the group algebra.

Then $S_n$ acts by left multiplication on $V_\lambda$. 

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**Theorem (Young, 1901)**

$(V_\lambda)_{\lambda \vdash n}$ forms a complete set of irreducible representations of $S_n$.

note: in fact, $\alpha_\lambda = \frac{\dim(V_\lambda)}{n!}$. 
Young symmetrizer (3/3)

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\((V_\lambda)_{\lambda \vdash n}\) forms a complete set of irreducible representations of \( S_n \).

note: in fact, \( \alpha_\lambda = \frac{\dim(V_\lambda)}{n!} \).

Next step: compute the trace.
Reformulation

Our goal

Let $\mu$ be a partition of $n$ and $\pi$ a permutation of cycle-type $\mu$. We want to compute the trace $\chi^\lambda_\mu$ of

$$\rho^\lambda(\pi) : \mathbb{C}[S_n]p_\lambda \rightarrow \mathbb{C}[S_n]p_\lambda$$

$$x \mapsto \pi \cdot x$$
Reformulation

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Problem: $\mathbb{C}[S_n]p_\lambda$ does not have an explicit basis
Reformulation

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Let $\mu$ be a partition of $n$ and $\pi$ a permutation of cycle-type $\mu$. We want to compute the trace $\chi_\mu^\lambda$ of

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$x \mapsto \pi \cdot x$

Problem: $\mathbb{C}[S_n]p_\lambda$ does not have an explicit basis

Lemma

$$\text{tr}(\rho^\lambda(\pi)) = \text{tr} \left( \begin{array}{c} \mathbb{C}[S_n] \rightarrow \mathbb{C}[S_n] \\ x \mapsto \pi \cdot x \cdot p_\lambda \end{array} \right)$$

Proof: $\mathbb{C}[S_n] = \mathbb{C}[S_n]p_\lambda \oplus \mathbb{C}[S_n](1 - p_\lambda)$ and the application $(x \mapsto \pi xp_\lambda)$ is $\rho^\lambda(\pi)$ on $\mathbb{C}[S_n]p_\lambda$ and 0 on $\mathbb{C}[S_n](1 - p_\lambda)$
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Lemma

$$\text{tr}(\rho^\lambda(\pi)) = \text{tr}\left(\mathbb{C}[S_n] \rightarrow \mathbb{C}[S_n] \quad x \mapsto \pi \cdot x \cdot p_\lambda\right)$$

Corollary

$$\chi_\mu^\lambda = \text{tr}(\rho^\lambda(\pi)) = \alpha_\lambda \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \varepsilon(\tau) \text{tr}(x \mapsto \pi \cdot x \cdot \sigma \cdot \tau)$$
Reformulation

Our goal

Let $\mu$ be a partition of $n$ and $\pi$ a permutation of cycle-type $\mu$. We want to compute the trace $\chi_\mu^\lambda$ of

$$\rho^\lambda(\pi) : \mathbb{C}[S_n]\rho_\lambda \rightarrow \mathbb{C}[S_n]\rho_\lambda$$

$$x \mapsto \pi \cdot x$$

Problem: $\mathbb{C}[S_n]\rho_\lambda$ does not have an explicit basis

Lemma

$$\text{tr}(\rho^\lambda(\pi)) = \text{tr} \left( \begin{array}{cc} \mathbb{C}[S_n] & \mathbb{C}[S_n] \\ x & \pi \cdot x \cdot \rho_\lambda \end{array} \right)$$

Corollary

$$\chi_\mu^\lambda = \text{tr}(\rho^\lambda(\pi)) = \alpha_\lambda \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \varepsilon(\tau) \sum_{g \in S_n} \delta_{\pi \cdot g} \cdot \sigma \cdot \tau \cdot g$$
First formula

\[
n! \frac{\text{tr}(\rho^\lambda(\pi))}{\dim(V_\lambda)} = \sum_{\sigma \in S_n} \delta_{\pi g \sigma \tau = g} \sum_{\tau \in CS(T_0)} \sum_{\sigma \in RS(T_0)} \varepsilon(\tau) \delta_{\pi g \sigma \tau = g}
\]
Case $\alpha = 1$ and Young symmetrizer

First formula

$$n! \frac{\text{tr}(\rho^\lambda(\pi))}{\dim(V_\lambda)} = \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \varepsilon(\tau) \sum_{g \in S_n} \delta_{\pi g \sigma \tau = g}$$

...(some combinatorial manipulations on sums)...

$$n! \frac{\chi^\lambda_\mu}{\dim(V_\lambda)} = \sum_{\sigma, \tau \in S_n} \varepsilon(\tau) F_{\sigma, \tau}(\lambda),$$

where

$$F_{\sigma, \tau}(\lambda) = \left\{ \text{fillings } T \text{ of } \lambda \text{ such that } \sigma \in \text{RS}(T), \tau \in \text{CS}(T) \right\}$$

Example for $\sigma = (1, 2) \in S_6, \tau = (1, 3) \in S_6$: filling $T = \begin{array}{ccc} 5 & 2 & 1 \\ 4 & 3 & 6 \end{array}$
Further simplifications

Reminder: \[ n! \frac{\chi_\mu}{\dim(V_\lambda)} = \sum_{\sigma, \tau \in S_n} \varepsilon(\tau) F_{\sigma, \tau}(\lambda). \]

We are interested in \( \chi_\mu^{\lambda n-k} \Rightarrow \) we can choose \( \pi \in S_k \subset S_n \).
Further simplifications

Reminder:

\[ n! \frac{\chi^\lambda_{\mu}}{\text{dim}(V_\lambda)} = \sum_{\sigma, \tau \in S_n} \varepsilon(\tau) F_{\sigma, \tau}(\lambda). \]

We are interested in \( \chi^\lambda_{\mu 1^n-k} \Rightarrow \) we can choose \( \pi \in S_k \subset S_n \).

Observation:

- terms vanish except for \( \sigma, \tau \) also in \( S_k \);
- for \( \sigma, \tau \) in \( S_k \),
  
  \[ F_{\sigma, \tau}(\lambda) = (n-k)! \tilde{N}_{\sigma, \tau}(\lambda), \]

where \( \tilde{N}_{\sigma, \tau}(\lambda) = \left| \left\{ \text{injective functions } f : \{1, \cdots, k\} \to \lambda \text{ such that } \sigma \in \text{RS}(f), \tau \in \text{CS}(f) \right\} \right| \)

Example for \( \sigma = (1, 2) \in S_3, \tau = (1, 3) \in S_3 \): filling \( T = \begin{array}{ccc} 2 & 1 & \text{ } \\ \text{ } & \text{ } & \text{ } \\ 3 & \text{ } & \text{ } \end{array} \)
Further simplifications

Reminder:
\[ n! \frac{\chi_\mu^\lambda}{\dim(V_\lambda)} = \sum_{\sigma, \tau \in S_n} \varepsilon(\tau) F_{\sigma, \tau}(\lambda). \]

We are interested in \( \chi_{\mu, 1}^{\lambda, n-k} \Rightarrow \) we can choose \( \pi \in S_k \subset S_n \).

Observation:
- terms vanish except for \( \sigma, \tau \) also in \( S_k \);
- for \( \sigma, \tau \) in \( S_k \),
  \[ F_{\sigma, \tau}(\lambda) = (n - k)! \tilde{N}_{\sigma, \tau}(\lambda), \]
  where \( \tilde{N}_{\sigma, \tau}(\lambda) = \left\| \left\{ \text{injective functions } f : \{1, \cdots, k\} \to \lambda \text{ such that } \sigma \in RS(f), \tau \in CS(f) \right\} \right\| \)

We obtain:
\[ \frac{n!}{(n - k)!} \frac{\chi_{\mu, 1}^{\lambda, n-k}}{\dim(V_\lambda)} = \sum_{\sigma, \tau \in S_k} \varepsilon(\tau) \tilde{N}_{\sigma, \tau}(\lambda), \]
Further simplifications

Reminder: \[ n! \frac{\chi^\lambda_{\mu}}{\dim(V_\lambda)} = \sum_{\sigma, \tau \in S_n, \sigma \tau = \pi} \varepsilon(\tau) F_{\sigma, \tau}(\lambda). \]

We are interested in \( \chi^\lambda_{\mu 1^{n-k}} \Rightarrow \) we can choose \( \pi \in S_k \subset S_n \).

We have obtained:

\[
\frac{n!}{(n-k)!} \frac{\chi^\lambda_{\mu 1^{n-k}}}{\dim(V_\lambda)} = \sum_{\sigma, \tau \in S_k, \sigma \tau = \pi} \varepsilon(\tau) \tilde{N}_{\sigma, \tau}(\lambda),
\]

where \( \tilde{N}_{\sigma, \tau}(\lambda) = \left| \left\{ \text{injective functions } f : \{1, \cdots, k\} \to \lambda \text{ such that } \sigma \in RS(f), \tau \in CS(f) \right\} \right| \)
Further simplifications

Reminder:

\[ n! \frac{\chi_{\mu}^\lambda}{\dim(V_\lambda)} = \sum_{\sigma,\tau \in S_n \atop \sigma \tau = \pi} \varepsilon(\tau) F_{\sigma,\tau}(\lambda). \]

We are interested in \( \chi_{\mu 1}^{\lambda n-k} \Rightarrow \) we can choose \( \pi \in S_k \subset S_n \).

We have obtained:

\[ \frac{n!}{(n-k)!} \frac{\chi_{\mu 1}^{\lambda n-k}}{\dim(V_\lambda)} = \sum_{\sigma,\tau \in S_k \atop \sigma \tau = \pi} \varepsilon(\tau) N_{\sigma,\tau}(\lambda), \]

where \( N_{\sigma,\tau}(\lambda) = \left\{ \text{functions } f : \{1, \cdots, k\} \to \lambda \text{ such that } \sigma \in RS(f), \tau \in CS(f) \right\} \)

One can forget injectivity condition: non-injective functions have a total 0-contribution.
End of our proof

Theorem (F., Śniady 2007, conjectured by Stanley 2006)

\[
\frac{n!}{(n-k)!} \frac{\chi^\lambda_{\mu_1} \chi^{n-k}}{\dim(V_\lambda)} = \sum_{\sigma, \tau \in S_k, \sigma \tau = \pi} \varepsilon(\tau) N_{\sigma, \tau}(\lambda)
\]

Proof: the few previous slides!
Case $\alpha = 1$ and Young symmetrizer

End of our proof

Theorem (F., Śniady 2007, conjectured by Stanley 2006)

$$\text{Ch}_{\mu}^{(1)}(\lambda) = \sum_{\sigma, \tau \in S_k \atop \sigma \tau = \pi} \varepsilon(\tau) N_{\sigma, \tau}(\lambda)$$

Proof: the few previous slides!
End of our proof

Theorem (F., Śniady 2007, conjectured by Stanley 2006)

\[ (-1)^k \text{Ch}_{\mu}^{(1)}(\lambda) = \sum_{\sigma, \tau \in S_k} (-1)^{|C(\tau)|} N_{\sigma, \tau}(\lambda) \]

\(|C(\tau)|\): nombre de cycle de \(\tau\).
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Lemma

Let \(\sigma, \tau\) in \(S_k\). Then \(N_{\sigma, \tau}(\lambda(p, q))\) is a polynomial in \(p\) and \(q\) with non-negative integer coefficients and degree \(|C(\sigma)|\) in \(p\) and \(|C(\tau)|\) in \(q\).
End of our proof

**Theorem (F., Śniady 2007, conjectured by Stanley 2006)**

\[-1]^k \text{Ch}_{\lambda}^{(1)} = \sum_{\sigma, \tau \in S_k, \sigma \tau = \pi} (-1)^{|C(\tau)|} N_{\sigma, \tau}(\lambda)\]

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**Corollary**

\([-1]^k \text{Ch}_{\lambda}^{(1)}(\lambda(p, q))\) is a polynomial in \(p\) and \(-q\) with non-negative integer coefficients.
An example of $N_{\sigma,\tau}(\lambda(p,q))$

Let $\sigma = (1 2)$ and $\tau = \text{id}_2$. $N_{\sigma,\tau}(\lambda)$ count the number of ordered choice of two boxes of $\lambda$ in the same row.
Case $\alpha = 1$ and Young symmetrizer

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Recall that $\lambda(p, q) =$ .

![Diagram](image-url)
An example of $N_{\sigma, \tau}(\lambda(p, q))$

Let $\sigma = (1 2)$ and $\tau = \text{id}_2$.
$N_{\sigma, \tau}(\lambda)$ count the number of ordered choice of two boxes of $\lambda$ in the same row.

Recall that $\lambda(p, q) = \ldots$.

Hence

$$N_{(1 2), \text{id}_2}(\lambda(p, q)) = \sum_{i \geq 1} p_i q_i^2.$$
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Case $\alpha = 1$ and Young symmetrizer

Pair of permutations and graphs embedded in surfaces

There is a (classical) bijection between

$$S_k \times S_k \iff \left\{ \begin{array}{l} \text{bicolored graphs} \\
\text{embedded in orientable surfaces} \\
\text{with } k \text{ labelled edges.} \\
\end{array} \right. \right.$$ 

(up to isomorphism (with a slight technical condition)
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\[ \sigma = (1 \ 5 \ 2)(3 \ 4) \]
\[ \tau = (1 \ 2 \ 3 \ 5 \ 4) \]

- cycles of the product $\leftrightarrow$ “faces” of the map;
- $N_{\sigma,\tau}$ depends only on the underlying graph (neither on the embedding nor on edge multiplicities).
Case $\alpha = 1$ and Young symmetrizer

Stanley’s formula in terms of map

Theorem (F., Śniady 2007, conjectured by Stanley 2006)

$$(-1)^k \text{Ch}^{(1)}_{\mu}(\lambda) = \sum_{M \text{ bipartite oriented map of face-type } \mu} (-1)^{|V^\bullet(M)|} N_{G(M)}(\lambda)$$
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It is classical to count maps via characters of the symmetric group using **Frobenius counting formula** (Stanley, Jackson, Vinsenti, Jones, Zagier, Goupil, Schaeffer, Poulhalon).

But both formulas do not seem to be linked!
We just proved Lassalle’s conjecture for $\alpha = 1$. 
We just proved Lassalle’s conjecture for $\alpha = 1$.

**Theorem (F. Śniady, 2011)**

Lassalle’s conjecture holds also for $\alpha = 2$.

Next two slides:

- representation-theoretical interpretation of $\theta_{\mu}^{\lambda,(2)}$ (involves Gelfand pair);
- combinatorial formula for $\text{Ch}_{\mu}^{(2)}$. 
Definition of Gelfand pairs

Let $G$ be a finite group and $K$ a subgroup of $G$. We say that $(G, K)$ is a Gelfand pair if

- The induced representation $1^G_K$ is multiplicity free;
- or equivalently, the $\mathbb{C}[K \backslash G / K]$ is commutative

$\mathbb{C}[K \backslash G / K]$: subalgebra of $\mathbb{C}[G]$ formed by elements invariants by left and right multiplication by $k \in K$
Definition of Gelfand pairs

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Theory of Gelfand pairs extends representation theory of finite groups (RTFG).

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**Theorem (Stembridge, 1992)**

$\theta^{\lambda,(2)}_{\mu}$ are the zonal spherical values of the Gelfand pair $(S_{2n}, H_n)$ ($H_n$ is the hyperoctahedral group).
Combinatorial formula for $\text{Ch}_{\mu}^{(2)}$

Theorem (F., Śniady 2011)

$(-1)^k 2^{\ell(\mu)} \text{Ch}_{\mu}^{(2)}(\lambda) = \sum_{M \text{ bipartite non-oriented maps of face-type } \mu} (-2)^{|V(M)|} N_{G(M)}(\lambda)$
Case $\alpha = 2$ and Hecke algebra of $(S_{2n}, B_n)$

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Implies Lassalle’s conjecture for $\alpha = 2$.

There is a formula, analog to *Frobenius counting formula*, counting non-oriented maps using zonal spherical functions of $(S_{2n}, H_n)$ (Goulden, Jackson, 1996). But, once again, it does not seem related to our formula!
A combinatorial solution to the general case?

Conjecture (hope?)

There exists a weight $w_M(\alpha - 1)$, polynomial with non-negative coefficients in $\alpha - 1$, such that

$$(-1)^k \operatorname{Ch}_{\mu}^{(\alpha)}(\lambda) = \sum_{M \text{ bipartite non-oriented map of face-type } \mu} w_M(\alpha - 1) N_{G(M)}(\lambda)$$
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Goulden and Jackson (1996) have a similar conjecture for an extension of Frobenius counting formula. But still open!
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A partial result (Dołęga, F., Śniady, 2013)

There exist a combinatorial weight $w_M(\alpha - 1)$ such that, for any rectangular Young diagrams, the formula above holds.
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But this specific weight does not work in general (fails for $\mu = (9)$ and $\lambda$ non trivial superposition of 3 rectangles).
Conclusion and perspectives

- Still some weights to test…
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- Cases $\alpha = 1$ and 2 can be proved *a posteriori without* representation theory. So, if we *guess* the general combinatorial formula, there is some chance that we may *prove* it.
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- In any case, Jack polynomials are well-studied objects and a new combinatorial description would be welcome.
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- In any case, Jack polynomials are well-studied objects and a new combinatorial description would be welcome.

- From a combinatorial point of view, the conjecture suggest an interpolation between oriented and non-oriented framework: puzzling!