Linear coefficients of Kerov’s polynomials: bijective proof and refinement of Zagier’s result

Valentin Féray (LaBRI), Ekaterina Vassilieva (LIX)

LaBRI, cnrs, Université Bordeaux 1, F-33400 Talence, France - e-mail: feray@labri.fr  
LIX, cnrs, École Polytechnique, F-91128 Palaiseau, France - e-mail: ekaterina.vassilieva@lix.polytechnique.fr

Introduction

Question
Let \( m \leq N \) with \( m \equiv N \pmod{2} \). What is the number \( B(N, m) \) of permutations \( \sigma \) of size \( N \):

- with \( m \) cycles (notation : \( \kappa(\sigma) = m \) );
- such that \( (1 \, 2 \, \ldots \,

Motivations

- particular cases of coefficients of some character polynomials (ask for details!).
- surprising formula (Zagier [4], 1995):

\[
\frac{N(N+1)}{2}B(N, m) = \# \{ \sigma \in S_{N+1} \text{ avec } \kappa(\sigma) = m \} = A(N+1, m) \quad \text{(Stirling number)}
\]

Main result

- A combinatorial proof of (1) (asked by Stanley [3], 2009).
- A refinement taking the cycle type of permutations into account.

Reformulation using maps

Permutations \( \simeq \) Unicellular red-rooted bipartite maps
\[ \sigma = (1\, 2 \, 3 \, 4 \, 5\, 8), \quad (1 \, 2 \, \ldots \, N)^{\sigma^{-1}} = (1 \, 3 \, 5 \, 7 \, 6 \, 8) \]

\[\Rightarrow B(N, m) = \{ \text{rooted unicellular bipartite maps with } N \text{ edges and } \{ \text{red and blue vertices} \} \}
\]

\[A(N+1, m) = \{ \text{rooted unicellular bipartite maps with } N+1 \text{ edges and } m \text{ blue vertices} \} \]

Is there a bijection explaining equation (1)?
We didn’t manage to construct a direct one.

Blue-partitioned maps and star thorn trees

Equivalent statement

Definition 1. A blue-partitioned map is a map with a partition of its blue vertices.
- \( C(N, m) := \# \) rooted unicellular bipartite map with \( N \) edges and \( m \) blocks of blue vertices.
- \( D(N, m) := \# \) same objects with only one red vertex.

Proposition 2.
\[
\sum_p C(N, p)(x)_p = \sum_p A(N, m)x^m
\]
\[
\sum_p D(N, p)(x)_p = \sum_p B(N, m)x^m
\]

Remark 3. The refined version uses symmetric functions!
\( \Rightarrow \) Equation (1) is equivalent to:
\[
\forall p \leq N, \quad N(N+1)D(N, p) = C(N+1, p)
\]

Theorem 4 (Morales, V [2], 2009). The quantity \( C(N, p) \) is also the number of permuted star thorn trees i.e. bicolored trees with:
- only one red vertex (the root);
- \( p \) blue vertices;
- \( N - p \) thorns of blue (resp. red) extremity;
- a bijection between thorns of blue extremity and thorns of red extremity.

Corollary 5.
\[
(N+1-p)C(N+1, p) = N(N+1)C(N, p)
\]

Therefore, Equation (1) \( \Rightarrow (N+1-p)D(N, p) = C(N, p) \)

Combinatorial construction

From partitioned maps to permuted star thorn trees

Idea: merge vertices of the same block and cut all edges except one per block

Rules to merge vertices and choose which edges to keep:
1. Draw each vertex with its maximum as right-most edge and order them in decreasing order of their maxima (like in Foata’s transform).
2. Merge vertices and keep the left-most edge.

Proposition 6. Injective mapping (ask for a demo of the inverse).
- Its image can be characterized using auxiliary graphs:

Using this characterization, one can check that the proportion of the image is \( \frac{1}{1+\frac{1}{p}} \).

Remarks

- Another simpler combinatorial proof has been found recently in [1].
- Analogue results for maps on locally orientable surfaces?

References


VALMAT 2010 - 22nd international Conference on Formal Power Series & Algebraic Combinatorics - August 2-6, 2010 - San Francisco - United States