Application of graph combinatorics to rational identities of type A

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Definitions and examples

Motivations

The aim of this work is to study combinatorially the function

$$\Psi_G = \sum_{w \in \mathcal{L}(G)} \psi_w, \text{ where}$$

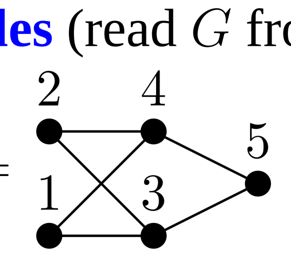
$$\psi_{1234\dots n} := \frac{1}{(x_1 - x_2)(x_2 - x_3)\cdots(x_{n-1} - x_n)},$$

G is an oriented graph and $\mathcal{L}(G)$ the set of its linear extensions.

This function appears in the following contexts:

- Greene (1992) has computed this function for some graphs to give a new proof of the Murnaghan-Nakayama formula.
- Ψ_G is the Laplace transform of the characteristic function of some pointed cone (current work with Victor Reiner).

Examples (read G from left to right)

- If $G =$  then $\Psi_G = \psi_{12345} + \psi_{12435} + \psi_{21345} + \psi_{21435} = \frac{x_1 \cdot x_2 - x_1 \cdot x_5 - x_2 \cdot x_5 - x_3 \cdot x_4 + x_3 \cdot x_5 + x_4 \cdot x_5}{(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_5)(x_4-x_5)}$,
- if G is disconnected then $\Psi_G = 0$,
- if G is acyclic then $\Psi_G = \frac{1}{\prod_{(i,j) \text{ edges of } G} (x_i - x_j)}$.

Some known results

For Hasse diagrams of posets, the denominator of the reduced Ψ_G is:

$$\prod_{(i,j) \text{ edges of } G} (x_i - x_j)$$

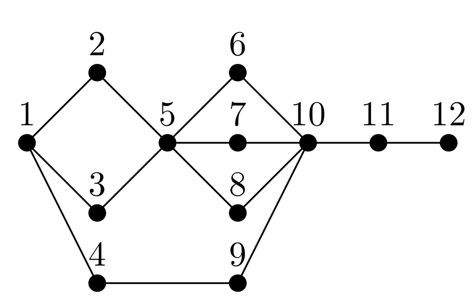
For a general class of graphs, Greene gives the following formula:

Theorem 1 (Greene, 1992) *If G is the Hasse diagram of a connected "strongly-planar" poset P, then*

$$\Psi_G = \prod_{y,z \in P} (x_y - x_z)^{\mu_P(y,z)}$$

where $\mu(x,y)$ denotes the Möbius function on the poset P.

For instance,

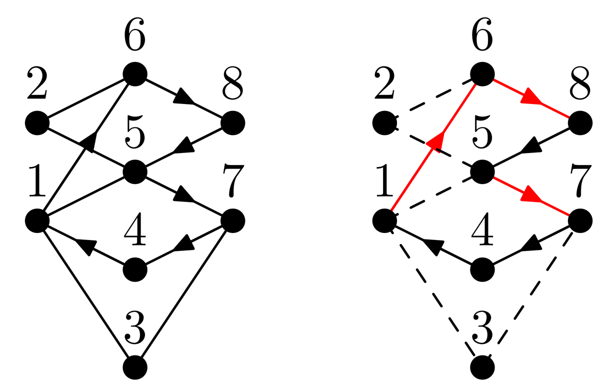
$G =$  $\Psi_G = \frac{(x_1 - x_5)(x_5 - x_{10})^2 (x_1 - x_{10})}{\prod_{(i,j) \in G} (x_i - x_j)}$

An inductive algorithm

Goal To compute $N(G) := \psi_G \cdot \prod_{\text{edge } (i,j)} (x_i - x_j)$.

An equality on linear extensions

Let G be an oriented graph with a cycle C.

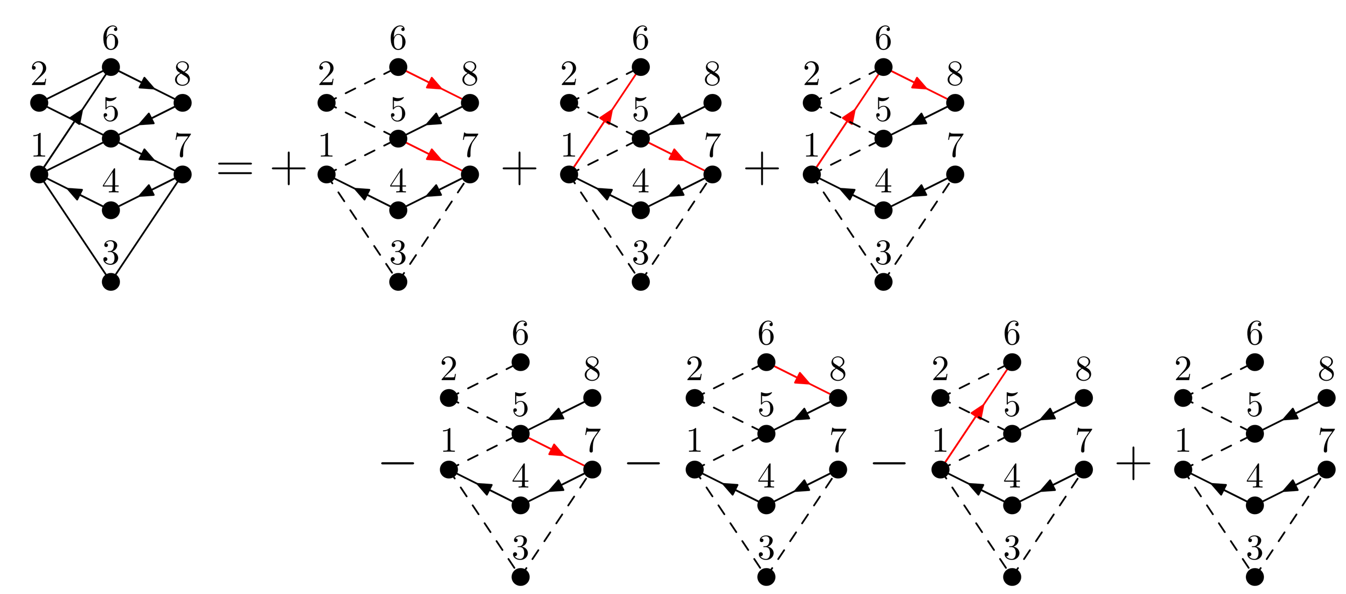


$C = 6-8-5-7-4-1-6$

The edges of C are not necessarily oriented as in G.

$LR(C) := \{$ edges (i,j) for which both orientations coincide $\}$.

Proposition 2



Corollary

Proposition 3

$$N(G) = \sum_{\substack{E' \subseteq LR(C) \\ E' \neq \emptyset}} \left[(-1)^{|E'|-1} N(G \backslash E') \prod_{(i,j) \in E'} (x_i - x_j) \right].$$

Iteration of this proposition $\Rightarrow \exists$ coefficients c_T in $\mathbb{Z}$ such that :

$$N(G) = \sum_{T \text{ subtree of } G} c_T \prod_{e \in E_G \backslash E_T} (x_{\alpha(e)} - x_{\omega(e)}).$$

Of course, the coefficients c_T depend on the chosen cycles C.

All the summands in the right-hand side are polynomials! This formula gives a **quicker way to compute** $N(G)$ (usually done by enumerating linear extensions).

Consequences

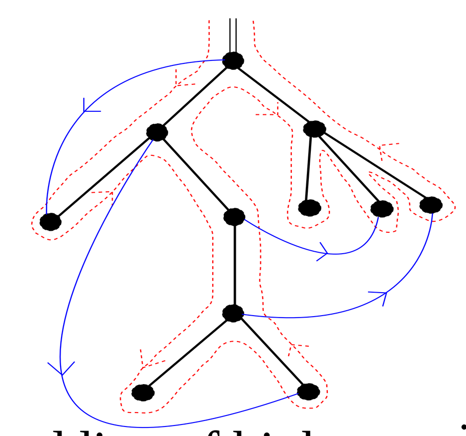
A combinatorial formula for $N(G)$

Suppose that G can be embedded in the plane. If we iterate the last proposition only for cycles C with counterclockwise orientation, we always obtain the same coefficients c_T. Moreover, they are either 0 or 1. So:

$$N(G) = \sum_{T \text{ s.t. } c_T = 1} \left[\prod_{(i,j) \in E_G \backslash E_T} (x_i - x_j) \right].$$

The coefficient of a tree T in G can be determined in this way:

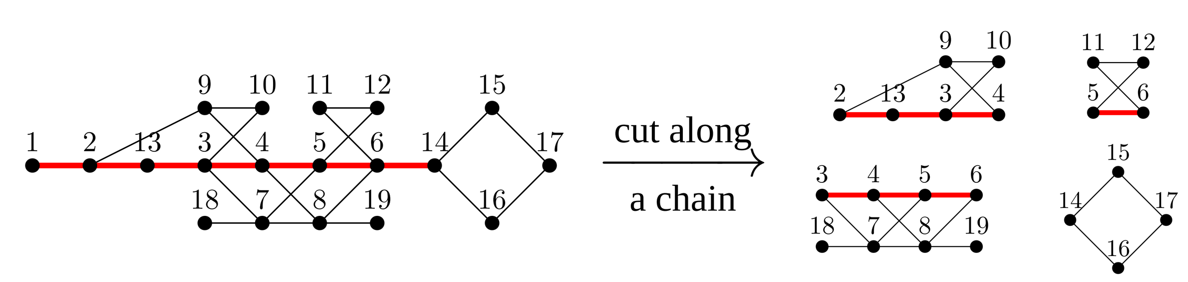
- fix a corner in the external face of G ;
- make the tour of the tree ;
- $c_T = 1$ iff, for any edge not in the tree, one crosses his first dart before the second one.



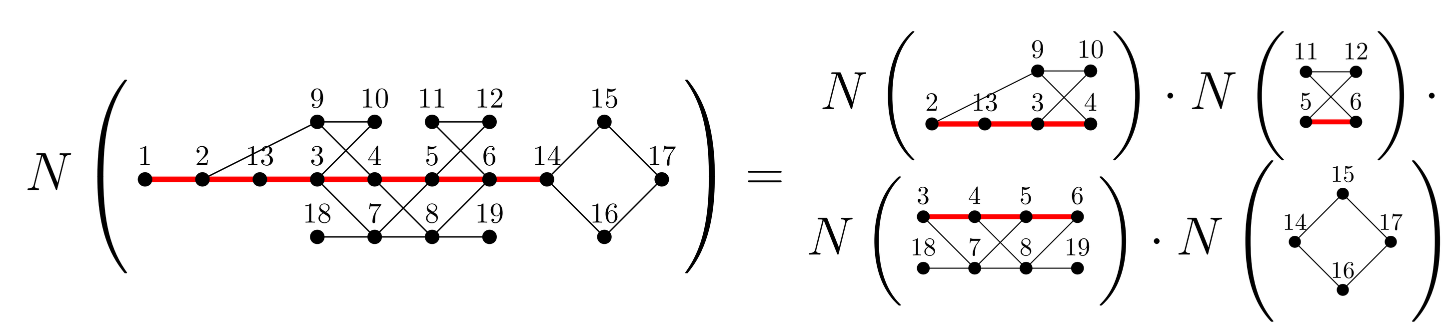
This **generalizes** to graphs with a rooted embedding of higher genus.

Chain factorization

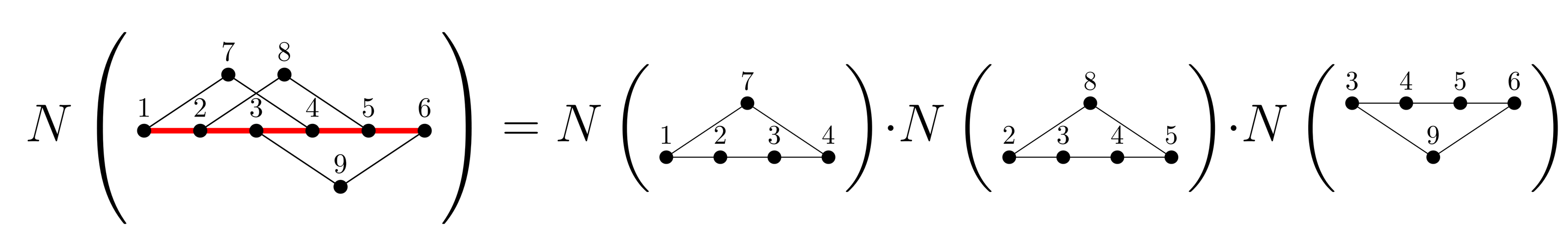
Let us cut a graph G into several pieces along a chain:



The numerator of the corresponding function can be factorized:



With this property, we can recover and **extend** Greene's theorem. For instance, Greene's formula is also true for the following poset:



References

[1] C. Greene, *A rational function identity related to the Murnaghan-Nakayama formula for the characters of $\mathcal{S}_n$*, J. Alg. Comb. **1** 3 (1992), 235-255.

[2] A.Boussicault. *Operations on posets and rational identities of type A*, FPSAC (2007).

Acknowledgments: The authors are grateful to A. Lascoux for his suggestion to work on these rational functions. The authors also thank J.G. Luque for useful discussions and T. Gomez Diaz for reviewing this poster.

FPSAC 2009 - 21st international Conference on Formal Power Series & Algebraic Combinatorics - July 20-24, 2007 - Hagenberg - Austria