Zeilberger's Algorithm – Creative Telescoping

Noam Szyfer
MAT 625 Seminar on Automatic Proofs of Binomial Sum Identities
University of Zurich, Spring Semester 2020

April 30, 2020

Contents

1 Introduction

1.1 Definite and indefinite sums

The main goal of this chapter is to study *definite* sums (in contrast to the previous chapter about Gosper's algorithm which focused on *indefinite* sums). Therefore it seems reasonable to first highlight the difference of these two notions of summation.

This difference turns out to be fully analogous to the perhaps more familiar contrast of *definite* vs. *indefinite* integration, which I will illustrate on the classical example of $\int e^{-x^2}dx$. Say one is set to find a closed form expression of $\int_0^a e^{-x^2}dx$, where a is an arbitrary positive[1] real number. If we could find a simple antiderivative $F(x)$ of e^{-x^2} we would be done since we then could write $\int_0^a e^{-x^2}dx = F(a) - F(0)$, however it is not advisably to go and try to find such antiderivative since it doesn't exist.

But still, the *definite* integral $\int_{-\infty}^{\infty} e^{-x^2}dx$ can be evaluated without the use of an antiderivative, and it has the rather nice value of $\sqrt{\pi}$. We see that sometimes even if we fail to compute an indefinite integral, there is still hope if we try to solve a definite integral with fixed boundaries. How does this translate to the notion of Summation?

For this, let us try to sum the binomial coefficient $\binom{n}{k}$, thought of for fixed n as a function of k. Gosper's algorithm will then tell us that this expression is not Gosper-summable, so the indefinite sum

[1] We use *positive* in the non-french way, meaning that a is *strictly* greater than zero.

$\sum_{k=0}^{K_0} \binom{n}{k}$, where K_0 is an arbitrary upper bound, cannot be expressed as simple hypergeometric terms in K_0 (and n). Nevertheless, the unrestricted sum $\sum_{k=0}^{\infty} \binom{n}{k}$, which at this point should be thought of as the indefinite integral $\int_{-\infty}^{\infty} e^{x^2} dx$ in our analogy, also has a perfectly nice simple form, as we know (or will also see in Example A_1) that $\sum_{k=0}^{\infty} \binom{n}{k} = 2^n$.

1.2 What exactly are we looking for?

Now, let's take a look at what exactly our goal is, and how to get there. Our main interest are sums of the form

$$f(n) = \sum_k F(n,k) \tag{1.2.1}$$

where $F(n,k)$ is a hypergeometric term in both arguments, i.e., $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are both rational functions of n and k.

We want to find a recurrence relation for the sum $f(n)$, and for that we will first find a recurrence relation for the summand $F(n,k)$. If this approach seems familiar, it is no coincidence because it is basically the same approach as we use with Sister Celine's algorithm. But, Zeilberger's Algorithm does the same job a great deal faster.

What kind of recurrence relation can we expect to find in general? It's true that if we find some nice hypergeometric function $G(n,k)$ such that

$$F(n,k) = G(n,k+1) - G(n,k) \tag{1.2.2}$$

we can use Gosper's Algorithm to compute $f(n)$. (We can actually do even more, since Gosper's Algorithm will let us express the sum as a function of a variable upper limit.) But as we have seen above, this does not always work, as in the example of the summand $F(n,k) = \binom{n}{k}$, so it is too much to expect to find a recurrence of this form in every case.

In Chapter 2 we saw that we sometimes (or indeed rather often[2]) we can find a $G(n,k)$ such that it is true that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{1.2.3}$$

If that happens, than, even though we can't compute the indefinite sum of $F(n,k)$, we can prove the definite summation identity $f(n) = const$ (as it is shown in chapter 2, page 25).

Alas, we also cannot expect to always find a $G(n,k)$ such that (1.2.3) is satisfied. But if we take a somewhat more general difference operator in n on the left-hand side of (1.2.3), we finally get a recurrence which we can prove to exist (under very general circumstances[3]). The recurrence that we can guarantee to exist is

$$\sum_{j=0}^{J} a_j(n) F(n+j,k) = G(n,k+1) - G(n,k), \tag{1.2.4}$$

where $a_i(n)$ is a polynomial in n for all $i \in \{0, ..., J\}$. If you're keen on using some colored highlighters in your work, this is definitely the moment to make use of one, since it is the central mission of the *method of creative telescoping* to produce the recurrence (1.2.4).

1.3 Notational convenience: The forward shift operator in n

Later, when we will employ Maple to find the recurrences that the summands $F(n,k)$ satisfy, the answer that is provided to us will be shown in the so-called *operator form*, which is an arguably more convenient way to write down recurrence relations. It works as follows:

Denote by N the forward shift operator in n. The notation of the recurrence relation

[2] We will hear more about the precise definition of "often" in chapter 7.

[3] Namely, that $F(n,k)$ is a non-zero proper hypergeometric term, see page 105 and page 64

$$a_0 f(n) + a_1 f(n+1) + ... + a_j f(n+j) = 0 \tag{1.3.1}$$

can then be written as

$$(a_0 + a_1 N + ... + a_j N^j) f(n) = 0 \tag{1.3.2}$$

where the a_i are arbitrary coefficients, which may and will include not only constants but also polynomials in n.

2 Application of the Algorithm

Before we will see the inner workings of the algorithm, as well as the proof that a nontrivial recurrence of the form (1.2.4) always exists, which will be done in the next two parts on this chapter, we will dedicate the remainder of this first part on some examples, which will also illustrate in detail how to use the Maple code that is provided on the website[4] of the book. We will further see that there are some instances in which we will not be able to find a definitive answer to the question whether there exists a closed form of our sum $f(n)$ with creative telescoping alone. These problems will be solved in chapter 8 with the help of Petkovšek's algorithm.

2.1 Three different cases

Suppose that we execute the *creative telescoping algorithm*, and we find a recurrence of the form (1.2.4), for the summand function $F(n,k)$, and a rational function $R(n,k)$ for which $G(n,k) = R(n,k)F(n,k)$. The coefficients on the left-hand side are independent of k, so we can sum (1.2.4) over all integer values of k and obtain

$$\sum_{j=0}^{J} a_j(n) f(n+j) = 0 \tag{2.1.1}$$

Note that the right-hand side telescopes to 0 when for instance $G(n,k)$ has compact support, which is often the case, at least in all further examples. It's important to notice that in some cases, there are apparent singularities in the certificate $G(n,k)$ for some k, i.e. the denominator vanishes for some specific values of k. However, oftentimes this is not that problematic, as we can get rid of the denominator by simple algebraic manipulations (seen in the examples in Scenario A). Another possibility is that the values that make the denominator vanish lie outside of the support of the original summand, so we are effectively not summing over these values of k. But there are cases where we need to be more careful, as we can see in the examples of Scenario B. A quick Google search reveals that this 'problem' of singularities is a topic of ongoing research which means that we won't go into full detail here.

2.2 Scenario A

If we get lucky, we are in the first case where $J = 1$, i.e., the equation (2.1.1) is a recurrence $a_0(n)f(n) + a_1(n)f(n+1) = 0$. Applying some simple algebraic manipulations then yields $f(n+1)/f(n) = -a_0(n)/a_1(n)$, which is a rational function of n. So we have found that $f(n)$ is indeed a hypergeometric term, namely

$$f(n) = f(0) \prod_{j=0}^{n-1} -\frac{a_0(j)}{a_1(j)}. \tag{2.2.1}$$

We might have to be a little careful with the selection of the 'initial value' $f(0)$. Since the recurrence is only guaranteed to hold when $F(n,k)$ is non-zero, we might have to 'shift' this value to $f(a)$ where a is the first value for which $F(a,k) \neq 0$. See Example A_4 for such a case.

Now we will take a look at some examples.

[4]https://www.math.upenn.edu/∼wilf/AeqB.html

2.2.1 Example A_1

As an easy introductory example, suppose we are interested in finding a closed form for the sum

$$f(n) = \sum_k \binom{n}{k}. \tag{2.2.2}$$

We start by using the provided Maple code to find the recurrence of the form (1.2.4). For that, we download the file 'EKHAD' from the homepage, move it to the main directory of maple (which can be found using the `currentdir()` command) and then type `read EKHAD` to load the algorithm. We proceed to define the summand $F(n,k)$ (not the sum $f(n)$), writing `F:=(n,k)-\> binomial(n,k)`.

To call for the *creative telescoping algorithm*, type now `ct(F(n,k),1,k,n,N)`, where `1` denotes the degree of the recurrence we are looking for (if no recurrence of this degree exists, Maple will return `0` telling us that we should try a higher order), `k` and `n` are respectively the summation and the running indices, and N denotes the forward shift operator which is used to print the recurrence that is satisfied (see 1.3). The answer Maple provides is

```
N-2, k/(-n+k-1)
```
(2.2.3)

Where we'll denote the rational function on the right by $R(n,k)$. The left-hand side then tells us, that for $G(n,k) = R(n,k)F(n,k)$ the following recurrence holds:

$$F(n+1,k) - 2F(n,k) = G(n,k+1) - G(n,k) \tag{2.2.4}$$

Notice that at this point there is no need to blindly trust the program to provide the correct recurrence. The rational function $R(n,k)$ that is provided by the algorithm can be used to directly verify the recurrence. This process is almost identical to the example found on page 25 in the book (in particular step 4), except that the *proof certificate* $R(n,k)$ is not exactly the same, as well as a slightly different recurrence relation. This is due to the fact that the example in the paper deals with the WZ proof algorithm, and not with the algorithm of creative telescoping. Nonetheless the verification is largely the same.

So we want to show that $F(n+1,k) - 2F(n,k) = G(n,k+1) - G(n,k)$. As a first step, we will take a closer look at the function $G(n,k) = R(n,k)F(n,k)$ to see if we can simplify it a bit.

$$G(n,k) = \frac{k}{-n+k-1}\binom{n}{k} = -\frac{kn!}{k!(n-k)!(n-k+1))} = -\frac{n!}{(k-1)!(n-(k-1))!} = -\binom{n}{k-1} \tag{2.2.5}$$

Now we wish to check if the recurrence 2.2.4 holds. Writing our functions explicitly, we obtain

$$\binom{n+1}{k} - 2\binom{n}{k} = -\binom{n}{k} + \binom{n}{k-1} \tag{2.2.6}$$

We have arrived at a so called *routinely verifiable* identity, which according to the paper your pet chimpanzee[5] could check if the equation is correct. Nonetheless, we'll do it ourselves. We replace all the binomial coefficients in sight by the quotient of factorials it represents, and cancel out all of the factorials by suitable divisions, until we only have a polynomial identity involving n and k.

$$\frac{(n+1)!}{k!(n+1-k)!} - \frac{2n!}{k!(n-k)!} = -\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

We multiply both sides with $\frac{k!(n-k+1)!}{n!}$ to obtain

$$\begin{gathered} n+1-2(n-k+1) = -(n-k+1)+k \\ \Leftrightarrow 2k-n-1 = 2k-n-1 \end{gathered}$$

which is obviously true, so our recurrence is correct.

Looking back now at the recurrence (2.2.4), by summing over all k we obtain

$$f(n+1) - 2f(n) = 0 \tag{2.2.7}$$

[5] albeit a mathematically rather adept chimpanzee, if you ask me!

and we can see that we are indeed in the case $J = 1$, with $a_0 = -2$ and $a_1 = 1$, as well as $f(0) = 1$ since $\binom{0}{k}$ vanishes for all $k \neq 0$ and is 1 if $k = 0$. So our result is

$$f(n) = f(0)\prod_{j=0}^{n-1} -\frac{a_0(j)}{a_1(j)} = f(n) = \prod_{j=0}^{n-1} -\frac{(-2)}{1} = \prod_{j=0}^{n-1} 2 = 2^n \tag{2.2.8}$$

2.2.2 Example A_2

Now we want to see the algorithm work in a slightly more interesting example. For this, we look at exercise 1 (b) in the back of the chapter. Specifically, we are interested in the sum

$$f(n) = \sum_k \binom{x}{k}\binom{y}{n-k} \tag{2.2.9}$$

for x and y arbitrary positive integers. We proceed as before and use Maple to provide us with the following output:

`(n+1)N+n-x-y,(y-n+k)k/(-n-1+k),` (2.2.10)

which informs us that the summand $F(n,k) := \binom{x}{k}\binom{y}{n-k}$ satisfies the recurrence

$$(n+1)F(n+1,k) + (n-x-y)F(n,k) = G(n,k+1) - G(n,k) \tag{2.2.11}$$

where $R(n,k) := \frac{(y-n+k)k}{-n-1+k}$ (the right-hand side of the maple output) and

$$G(n,k) := R(n,k)F(n,k) = -k\binom{x}{k}\binom{y}{n-k+1}. \tag{2.2.12}$$

Maybe it would be a bit too bold to just trust the computer this time, but since the procedure of verifying the recurrence was more than a little tedious last time, there must be a better way to do it, right? Since Mable is capable of symbolic computation, there is a way to let Maple do the Job. Since all we did was to expand all terms, collect and simplify, we can let the Computer do the deed. For this we can type

`simplify(expand((n+1)F(n+1,k)+(n-x-y)(F(n,k)-G(n,k+1)+G(n,k)))` (2.2.13)

into Maple. If our recurrence is correct, it will return `0`, which luckily is the result here. Again, summing over all k we can see that we are indeed in the case $J = 1$, as well as $f(0) = 1$ since for all non-zero k one of the binomial terms is 0, and for $k = 0$ we have $1 \cdot 1 = 1$. So our result is given by

$$\begin{aligned} f(n) = f(0)\prod_{j=0}^{n-1} -\frac{a_0(j)}{a_1(j)} &= \prod_{j=0}^{n-1} \frac{x+y-j}{j+1} \\ &= \frac{(x+y)(x+y-1)...(x+y-(n-1))}{n!} = \frac{(x+y)!}{n!(x+y-n)!} = \binom{x+y}{n} \end{aligned} \tag{2.2.14}$$

A small remark for the interested: This identity is known as *Vandermonde's identity.*

2.2.3 Example A_3

Another example we'll take a look at is exercise 1 (e). We want to know a closed form for

$$\sum_k (-1)^k \binom{n-k}{k} 2^{n-2k} \tag{2.2.15}$$

The creative telescoping algorithm delivers

`(n+1)N-n-2,-4(-n-1+k)k/(-n-1+2k)` (2.2.16)

informing us that the demanded recurrence for the summand $F(n,k) := (-1)^k\binom{n-k}{k}2^{n-2k}$ is given by

$$(n+1)F(n+1,k) - (n+2)F(n,k) = G(n,k+1) - G(n,k) \tag{2.2.17}$$

where again we denote the right hand side of the maple output by $R(n,k)$ and further

$$G(n,k) := R(n,k)F(n,k) = (-1)^{k+1}\binom{n-k+1}{k}2^{n-2k}. \tag{2.2.18}$$

One can check, either by hand or with the same method as used in example A2, that this recurrence is indeed satisfied. Again, we sum over k and see that $J = 1$ and further we have that $f(0) = 1$ since either the top or the bottom part of the binomial coefficient is negative, annihilating the hole summand except for when $k = 0$ at which point the value is 1. We then have the result

$$f(n) = f(0)\prod_{j=0}^{n-1} -\frac{a_0(j)}{a_1(j)} = \prod_{j=0}^{n-1}\frac{j+2}{j+1} = \frac{(n+1)!}{n!} = n+1 \tag{2.2.19}$$

2.2.4 Example A_4

Now we'll take a look at exercise 1 (a). We investigate the sum

$$\sum_k (-1)^k \binom{n}{k}\binom{2n-2k}{n+a} \tag{2.2.20}$$

for arbitrary non-negative a. We notice right away that $F(n,k) = 0$ for $n < a$. Indeed, we then have $n + a > 2n - 2k$, so the binomial coefficient on the right is 0. Now, giving this sum to the creative telescoping algorithm yields

```
2n+2+(a-n-1)N, 8(n+1)(-n+k-1/2)k/(-n-1+2k+a)(n+1+a)        (2.2.21)
```

therefore we have $R(n,k) = \frac{8(n+1)(-n+k-\frac{1}{2})k}{(-n-1+2k+a)(n+1+a)}$ and also

$$G(n,k) = R(n,k)F(n,k) = (-1)^k\binom{n}{k}\frac{4k}{n+1+a}\binom{2n-2k+1}{n+a} \tag{2.2.22}$$

and we are provided with the following recurrence relation:

$$(a-n-1)F(n+1,k) + (2n+2)F(n,k) = G(n,k+1) - G(n,k). \tag{2.2.23}$$

So we have again the case that $J = 1$. Now, we can again sum over all k to make the right-hand side telescope to zero since it has compact support. As noted before, $f(n) = 0$ for $n < a$. Further we have $f(a) = 1$, since the only surviving term is when $k = 0$, otherwise the binomial on the right is zero, and for $k = 0$ the summand is clearly equal to 1. All this gives us the result

$$f(n) = f(a)\prod_{j=a}^{n-1} -\frac{a_0(j)}{a_1(j)} = \prod_{j=a}^{n-1} 2\frac{j+1}{j+1-a} = \\ 2^{n-a}\frac{n(n-1)...(a+1)}{(n-a)(n-a-1)...\cdot 2\cdot 1} = 2^{n-a}\frac{n!}{a!(n-a)!} = 2^{n-a}\binom{n}{a} \tag{2.2.24}$$

This closed form also holds for $n < a$, since then the binomial coefficient is zero anyway. An interesting fun fact: This formula gives the number of a-dimensional faces of an n-dimensional hypercube[6].

2.3 Scenario B

Now we also might have a little less luck and find ourselves in a position with a recurrence of the form (2.1.1) where $J > 1$. But not all is lost. If the recurrence happens to have *linear* coefficients, i.e. the coefficients are only *constant* polynomials (in n), we are still able to solve the recurrence, since we have tools to solve linear recurrence relations. We will see this in the following examples.

[6]looked up in the on-line encyclopedia of integer sequences, *http://oeis.org*

2.3.1 Example B_1

Suppose we want to investigate the sum

$$f(n) = \sum_{0 \leq k \leq n/2} \frac{n}{n-k} \binom{n-k}{k} \quad , \; n \geq 1. \tag{2.3.1}$$

Once again we use the creative telescoping algorithm, to find a recurrence satisfied by $F(n,k)$. Maple returns

```
N^2-N-1, (-n+k)k/(-n-1+2k)(-n-2+2k)
```
(2.3.2)

so we have $R(n,k) = \frac{(-n+k)k}{(-n-1+2k)(-n-2+2k)}$ as well as

$$G(n,k) = R(n,k)F(n,k) = -\frac{n}{n-2(k-1)} \binom{n-k}{k-1} \tag{2.3.3}$$

and the following recurrence that is satisfied:

$$F(n+2,k) - F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{2.3.4}$$

Now we'd like to proceed as before, summing over all k and get a recurrence for $f(n)$. But we have to be a little more careful here, since we have a denominator in the function $G(n,k)$ which has the potential to spoil the game. If n is odd, there's no problem since $2(k-1)$ is even, so the denominator doesn't vanish for any k. In this case, we can proceed as usual. But what if n is even? Then there is a problem when $k = \frac{n}{2} + 1$. This is a real obstacle, since we want to sum the recurrence (2.3.4) (at least) for $0 \leq k \leq \frac{n+2}{2} = \frac{n}{2} + 1$. One way to proceed would be to close our eyes, sum the recurrence over, say, $0 \leq k \leq n-1$ and declare that the right-hand side telescopes to 0, ignoring the fact that somewhere in the middle of the telescope we divided briefly by zero.[7].

But there is also a more rigorous way to do it. Let us sum the recurrence (2.3.4) for all values of k which do not produce a singularity, i.e. $0 \leq k \leq \frac{n}{2} - 1$, and thus obtain

$$\sum_{k=0}^{\frac{n}{2}-1} F(n+2,k) - \sum_{k=0}^{\frac{n}{2}-1} F(n+1,k) - \sum_{k=0}^{\frac{n}{2}-1} F(n,k) = G(n,\frac{n}{2}) - G(n,0) \tag{2.3.5}$$

We first notice that $G(n,0) = 0$. Also, we have the following three equalities:

$$\sum_{k=0}^{\frac{n}{2}-1} F(n+2,k) = f(n+2) - F(n+2,\frac{n}{2}+1) - F(n+1,\frac{n}{2}) \tag{2.3.6}$$

$$\sum_{k=0}^{\frac{n}{2}-1} F(n+1,k) = f(n+1) - F(n+1,\frac{n}{2}) \tag{2.3.7}$$

$$\sum_{k=0}^{\frac{n}{2}-1} F(n,k) = f(n) - F(n,\frac{n}{2}) \tag{2.3.8}$$

Substituting these into (2.3.5) yields

$$f(n+2) - f(n+1) - f(n) = G(n,\frac{n}{2}) + F(n+2,\frac{n}{2}+1) + F(n+1,\frac{n}{2}) - F(n+1,\frac{n}{2}) - F(n,\frac{n}{2}), \tag{2.3.9}$$

so all we need to do at this point is to verify that the right-hand side is equal to zero. Again, this sounds like a job for our beloved pet chimpanzee, which will tell us that this is indeed the case. So we have verified that the recurrence

$$f(n+2) - f(n+1) - f(n) = 0 \tag{2.3.10}$$

holds for both odd and even n. At first glance we're screwed now, since $J > 1$. Should all our effort have been in vain? But no, there is hope! The recurrence we have found has *linear* coefficients, meaning that we know how to find an explicit formula for the recursion.

[7] Too see this method in action, see Example 6.1.1 in the book

The characteristic polynomial of this homogeneous linear recurrence is given by $x^2 - x - 1$, which has roots $\alpha_1 = \frac{1-\sqrt{5}}{2}$ and $\alpha_2 = \frac{1+\sqrt{5}}{2}$. So the general solution is given by $f(n) = c_1\alpha_1^n + c_2\alpha_2^n$. To find the constants c_1 and c_2 we have to inspect what the values are of $f(1)$ and $f(2)$. We can see that

$$f(1) = \sum_{0 \le k \le n/2} F(1,k) = \frac{1}{1-0}\binom{1-0}{0} = 1 \cdot 1 = 1 \tag{2.3.11}$$

$$f(2) = \sum_{0 \le k \le n/2} F(2,k) = \frac{2}{2-0}\binom{2-0}{0} + \frac{2}{2-1}\binom{2-1}{1} = 1 \cdot 1 + 1 \cdot 2 = 3 \tag{2.3.12}$$

Hence we need to solve the following system

$$\begin{cases} c_1\alpha_1 + c_2\alpha_2 = 1 \\ c_1\alpha_1^2 + c_2\alpha_2^2 = 3 \end{cases} \tag{2.3.13}$$

which we will once again happily give to our already trained pet chimpanzee. He will tell us that $c_1 = c_2 = 1$, therefore the recurrence is given by

$$f(n) = \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^n, \; n \ge 1 \tag{2.3.14}$$

This expression looks eerily familiar to the closed form of the Fibonacci Numbers. Indeed, this equation gives us the *Lucas Numbers*[8], which are closely related to the Fibonacci Numbers in that it is also attained through adding the last two terms of the sequence, which we have seen in our recurrence (2.3.10), but instead of taking the starting values $f(0) = 1$ and $f(1) = 2$ (as we would for the Fibonacci numbers) we take $f(0) = 1$ and $f(1) = 3$ instead. Analogously to the Fibonacci Numbers, it is somewhat surprising that we have a sequence of integers whose closed form involves the irrational number $\sqrt{5}$.

2.3.2 Example B_2

Here's another situation where the same procedure is used. Say we are looking for a closed form of the sum

$$f(n) = \sum_{0 \le k \le n/2} \binom{n}{2k} 4^k \tag{2.3.15}$$

Once again we give the summand $F(n,k) = \binom{n}{2k}4^k$ to Maple, which returns

$$\texttt{N\^{}2-2*N-3,(-4*k\^{}2+2*k)/((-n-1+2*k)*(-n-2+2*k))} \tag{2.3.16}$$

informing us that the sought after recurrence is given by

$$F(n+2,k) - 2F(n+1,k) - 3F(n,k) = G(n,k+1) - G(n,k) \tag{2.3.17}$$

with $R(n,k) = \frac{-4k^2+2k}{(-n-1+2k)(-n-2+2k)}$ as well as

$$G(n,k) = R(n,k)F(n,k) = \frac{4^k(1-2k)}{n-2(k-1)}\binom{n}{2k-1}. \tag{2.3.18}$$

We are in exactly the same situation as in Example B2: There is a denominator in $G(n,k)$ which potentially produces a singularity. But we can proceed exactly the same as before. For n odd there's no problem, and when n is even we sum the recurrence (2.3.17) for $0 \le k \le \frac{n}{2} - 1$, obtaining

$$f(n+2) - 2f(n+1) - 3f(n) = G(n,\frac{n}{2}) + F(n+2,\frac{n}{2}+1) + F(n+2,\frac{n}{2}) + 2F(n+1,\frac{n}{2}) + 3F(n,\frac{n}{2}) \tag{2.3.19}$$

where it is once again a routine verification that the right-hand side vanishes, giving us the recurrence

$$f(n+2) - 2f(n+1) - 3f(n) = 0 \tag{2.3.20}$$

[8]Often, the Lucas numbers are given the initial value $f(0) = 2$, which is a small difference to the sum we have here.

We are again in the case where $J > 1$, but the coefficients are constant, so there is no need to panic.

The characteristic polynomial of this recurrence is $x^2 - 2x - 3 = (x+1)(x-3)$, which has roots $\lambda_1 = -1$ and $\lambda_2 = 3$. Therefore the general solution is given by $f(n) = c_1(-1)^n + c_2 3^n$. Now to find the constants c_1 and c_2, we will inspect $f(0)$ and $f(1)$. We have

$$f(0) = \sum_k F(0,k) = \binom{0}{0} 4^0 = 1 \tag{2.3.21}$$

$$f(1) = \sum_k F(1,k) = \binom{1}{0} 4^0 = 1 \tag{2.3.22}$$

Hence, we need to solve the following system

$$\begin{cases} c_1 + c_2 & = 1 \\ -c_1 + 3c_2 & = 1 \end{cases} \tag{2.3.23}$$

which yields that $c_1 = c_2 = \frac{1}{2}$, therefore our recurrence is given by

$$f(n) = \frac{1}{2}((-1)^n + 3^n) \tag{2.3.24}$$

2.4 Scenario C

Even though we conquered all the challenges until now, there are circumstances in which the creative telescoping algorithm can only provide us with a recurrence relation, and we need the help of further machinery (explicitly, of Petkovšek's algorithm presented in chapter 8) to find a closed form solution of our recurrence, if it exists. This is illustrated in the following example.

2.4.1 Example C_1

Suppose we have the explicit formula for the Legendre polynomials

$$P_n(x) = 2^{-n} \sum_k (-1)^k \binom{2n-2k}{n-k} \binom{n-k}{k} x^{n-2k} \tag{2.4.1}$$

And we want to find a recurrence formula for these polynomials. Notice that the algorithm works exactly the same as before, even though the summand now contains the indeterminate x, which is allowed by the definition of 'proper hypergeometric'. If we give the summand to the maple program, we obtain

```
(n+2)N^2-(2n+3)Nx+n+1, 4(n+1)x^2(-n+k-1/2)k/((-n-1+2k)(-n-2+2k))      (2.4.2)
```

with $R(n,k) = \frac{4(n+1)x^2(-n+k-\frac{1}{2})k}{(-n-1+2k)(-n-2+2k)}$ as well as

$$G(n,k) = R(n,k)F(n,k) = 2^{-n}(-1)^{k+1} 2(n+1) \binom{2n-2k+1}{n-k+1} \binom{n-k+1}{n-2k+2} x^{n-2k+2}, \tag{2.4.3}$$

and

$$(n+2)F(n+2,k) - x(2n+3)F(n+1,k) + (n+1)F(n,k) = G(n,k+1) - G(n,k) \tag{2.4.4}$$

Summing over all k leads to the following recursion formula:

$$(n+2)P_{n+2}(x) - x(2n+3)P_{n+1}(x) + (n+1)P_n(x) = 0. \tag{2.4.5}$$

By shifting the value of n and slightly rearranging, we obtain the well-known *Bonnet's Recursion Formula*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \tag{2.4.6}$$

Since the coefficients in this recursion are not linear, there is nothing more to do for us in the moment.

3 What remains to be seen

We have seen how to apply the creative telescoping algorithm in various ways. What remains to be seen is the proof that we can always find a recurrence of the form (1.2.4), as well as a thorough investigation of how exactly the algorithm that is implemented in the available program works. These two topics will be examined by my colleagues Alain Schmid and Bettina Wohlfender.