Sister Celine's method for binomial sums and generalizations

Student seminar on automatic proofs of binomial identities

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Sister Celine

A picture of Sister Celine by Herbert Wilf

Mary Celine Fasenmyer was an American mathematician known mostly for her work in linear algebra and on hypergeometric sums.

Sister Celine

In her Ph.D. thesis at the University of Michigan in 1945, she developed a method for systematically finding recurrence relations for (sums of) hypergeometric polynomials.

Before her work, finding such a recurrence relation was a question of trial and error involving the use of various "algebraic tricks".

Her methods were capable of finding recurrence relations for a specific subset of generalized hypergeometric polynomials including, among others, the Legendre- and Jacobi-polynomials.

Sister Celine

In her second publication, she illustrated her general method by going back to results from her Ph.D.

Only in 1978 did Doron Zeilberger fall back on Sister Celine's "method". In my text I will also be following a part of his paper *Sister Celine's Technique and Its Generalizations*. Her method (or extensions thereof, such as Zeilberger's algorithm which will also be the topic of some seminar talks) is now used by computer algebra systems such as Mathematica and Maple.

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Multi-hypergeometric functions

Definition

A function $F : \mathbb{Z}^2 \to \mathbb{C}$ is called *multi-hypergeometric* if and only if there exist non-zero polynomials $P_1, Q_1, P_2, Q_2 : \mathbb{Z}^2 \to \mathbb{C}$ such that for all $(n, k) \in \mathbb{Z}^2$, we have

$$\begin{aligned} P_1(n,k)F(n,k) - Q_1(n,k)F(n-1,k) &= 0 \quad \text{and} \\ P_2(n,k)F(n,k) - Q_2(n,k)F(n,k-1) &= 0. \end{aligned} \tag{1}$$

Additionally, define:

$$\chi_S(x) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ if } x \notin S \end{cases}.$$

Multi-hypergeometric functions

Example

The function $F(n, k) = 2^k \chi_{[0,n]}(k)$ is multi-hypergeometric:

Take $Q_1(n, k) = P_1(n, k)$ and $Q_2(n, k) = 2P_1(n, k)$ to be any non-zero polynomials which are 0 if $k = n$.

Another example is $F(n, k) = \chi_{[0,n]}(k) \cdot (c + kq)$ for any constants $c, q \in \mathbb{R}$ (this corresponds to the arithmetic series).

Multi-hypergeometric functions

Example

Any product of binomial coefficients $\binom{n}{k} = \chi_{[0,n]}(k) \cdot \frac{n!}{k!(n-k)!}$ is multi-hypergeometric. For example, for any $r > 0$, the function $F(n,k) = \binom{n}{k}^r$ satisfies

$$\begin{aligned} n^r F(n,k) - (n-k)^r F(n,k) = 0 \quad \text{and} \\ k^r F(n,k) - (n-k+1)^r F(n,k-1) = 0. \end{aligned} \tag{2}$$

Sums that are of interest

We are looking for a simplification of the sum

$$G(n) = \sum_{k=-\infty}^{\infty} F(n,k), \tag{3}$$

where $F : \mathbb{Z}^2 \to \mathbb{C}$ is multi-hypergeometric.

Sister Celine's Method provides tools, described in the next section, that find a linear recurrence relation satisfied by $G(n)$. These tools are guaranteed to work if the terms $F(n,k)$ are of a special form (namely if they are proper hypergeometric, I will come back to this in two sections).

Outline of the method

The idea of the algorithm is to first find a linear recurrence relation for the term F, such that the coefficients of this recurrence *do not rely on* k. Then, we sum this recurrence over all integers k in order to get a recurrence for G. In a final step (which is not part of Sister Celine's method), we solve this recurrence in order to obtain a simpler closed form for G.

Linear recurrence for the sum

We are interested in finding a constant C and polynomials $P_0, \ldots, P_C$ such that P_0 and P_C are not identically 0 and

$$\sum_{i=0}^{C} P_i(n) G(n-i) = 0. \tag{4}$$

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The recurrence

We will be looking for non-negative integers M, N and coefficients $a_{r,s} : \mathbb{Z} \to \mathbb{C}$ for $r, s \in \mathbb{Z}$ such that

$$\sum_{r=0}^{M} \sum_{s=0}^{N} a_{r,s}(n) F(n-r, k-s) = 0 \tag{5}$$

for all $n, k \in \mathbb{Z}$.

It is important to note that the $a_{r,s}$ are independent of k, for reasons seen later.

Later, I will give some quantitative estimates for the choice of M, N.

How to find the recurrence

Obtaining such a recurrence for given M, N is fairly straight-forward: Since F is multi-hypergeometric, we can write the recurrence in the form (where n,k are such that $F(n,k) \neq 0$)

$$\sum_{r=0}^{M} \sum_{s=0}^{N} a_{r,s}(n) \quad \underbrace{\frac{F(n-r,k-s)}{F(n,k)}}_{=\text{quotient of two polynomials}} \quad = 0. \tag{6}$$

The left-hand side can be put over a single common denominator. The numerator is now a polynomial in n and k so we can collect powers of k. Setting all coefficients of k to 0, we get a system of linear equations for each n where the unknowns are the $a_{r,s}(n)$.

How to find the recurrence

If the only solution to this linear equation system for some n is $a_{r,s}(n) = 0$ for all r, s, then we have to try again with bigger constants M, N. Otherwise, we have succesfully found a (non-degenerate) recurrence relation for the $F(n, k)$

Pascal's rule as an example

Example

Consider again the term $F(n,k) = \binom{n}{k}$. Then for $0 \le k \le n$, we have $\frac{F(n-1,k)}{F(n,k)} = \frac{n-k}{n}$ and $\frac{F(n,k-1)}{F(n,k)} = \frac{k}{1-k+n}$. By performing the above steps for $M = N = 1$, we obtain Pascal's rule

$$F(n,k) - F(n-1,k) - F(n-1,k-1) = 0. \qquad (7)$$

Recurrence of summand $\rightarrow$ Recurrence of sum

After we found a recurrence relation for the $F(n, k)$, we can quickly find a recurrence relation for the sum $G(n)$ with the following useful Lemma:

Lemma

If F is multi-hypergeometric such that $G(n) = \sum_{k=-\infty}^{\infty} F(n, k)$ is well-defined for all n and if F satisfies a recurrence of the form (5), then $G(n)$ satisfies the linear recurrence relation

$$\sum_{r=0}^{M} G(n-r) \cdot \left(\sum_{s=0}^{N} a_{r,s}(n) \right) = 0 \quad \textit{for all } n \in \mathbb{Z}. \tag{8}$$

Proof of Lemma

$$
\begin{aligned}
0 &= \sum_{k=-\infty}^{\infty} \overbrace{\sum_{r=0}^{M}\sum_{s=0}^{N} a_{r,s}(n)F(n-r,k-s)}^{0} \\
&= \sum_{r=0}^{M}\sum_{s=0}^{N} a_{r,s}(n) \overbrace{\sum_{k=-\infty}^{\infty} F(n-r,k-s)}^{G(n-r)} \\
&= \sum_{r=0}^{M}\sum_{s=0}^{N} a_{r,s}(n)G(n-r) \\
&= \sum_{r=0}^{M} G(n-r)\cdot\left(\sum_{s=0}^{N} a_{r,s}(n)\right).
\end{aligned}
\tag{9}
$$

Back to Pascal's rule

Example

Coming back to Pascal's rule, let us consider the sum

$$G(n) = \sum_{k=-\infty}^{\infty} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}. \tag{10}$$

We already know that the summand satisfies the recurrence

$$F(n,k) - F(n-1,k) - F(n-1,k-1) = 0.$$

From Lemma 5, we thus obtain that $G(n) = 2G(n-1)$ and hence from $G(0) = 1$ that $G(n) = 2^n$. (Faster is the smart observation that $G(n) = (1+1)^n$ by the Binomial Theorem.)

Sum of squared binomials

Example

Consider the sum with summands $F(n,k) = \binom{n}{k}^2$. Applying the procedure from 1 for $M = N = 2$, we get the recurrence (5) with

$$a_{0,0}(n) = n, a_{1,0}(n) = a_{1,1}(n) = 1 - 2n, \tag{11}$$

$$a_{2,0}(n) = a_{2,2}(n) = n - 1, a_{2,1}(n) = 2 - 2n. \tag{12}$$

Using Lemma 5, we obtain

$$nG(n) + (2 - 4n)G(n-1) = 0, \tag{13}$$

so for $n \neq 0$,

$$G(n) = \frac{4n-2}{n} G(n-1). \tag{14}$$

This, together with $G(0) = 0$, implies that $G(n) = \binom{2n}{n}$.

Closed forms

Suppose that we found, for example using the methods from the previous slides, a recurrence of the form

$$\sum_{r=0}^{M} P(r, n) G(n-r) = 0. \tag{15}$$

We are now interested in finding a *closed form* for $G(n)$, i.e. an expression of the type $G(n) = \binom{2n}{n}$.

There is an algorithmic approach called *algorithm hyper*, which will also be treated within this seminar.

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The fundamental Theorem

In this section, I will try to give some quantitative estimates on when to expect the approach of the previous section to work, and what M, N need to be chosen. This section is following the fourth chapter of the great book $A{=}B$.

Proper hypergeometric functions

Definition

A function $F : \mathbb{Z}^2 \to \mathbb{C}$ is called *proper hypergeometric* iff there exist constants $U, V \in \mathbb{N}$, $a_i, b_i, c_i, u_j, v_j, w_j \in \mathbb{Z}$ for $i = 1, \dots, U$ and $j = 1, \dots, V$, $x \in \mathbb{C}$ and a polynomial $P : \mathbb{Z}^2 \to \mathbb{C}$ such that

$$F(n,k) = P(n,k) \cdot \frac{\prod_{i=1}^{U}(a_i n + b_i k + c_i)!}{\prod_{i=1}^{V}(u_i n + v_i k + w_i)!} x^k \qquad (16)$$

for all $n, k \in \mathbb{Z}$.

Remark

(16) is well-defined only if $a_i n + b_i k + c_i \geq 0$ for all i. Also, we set $F(n,k) = 0$ if $u_i n + v_i k + w_i < 0$ for any i.

Proper hypergeometric functions

Example

The function

$$F(n,k) = \binom{n}{k} 2^k = \frac{n!2^k}{k!(n-k)!} \tag{17}$$

is proper hypergeometric. So is the function

$$F(n,k) = \frac{1}{d_1 n + d_2 k + d_3 + 1} = \frac{(d_1 n + d_2 k + d_3)!}{(d_1 n + d_2 k + d_3 + 1)!} \tag{18}$$

for any $d_1, d_2, d_3 \in \mathbb{N}$.

Proper hypergeometric functions

Remark

The ideas described here arose initially in the original paper by Sister Celine, *A Note on Pure Recurrence Relations*, from 1949, where she was studying quote "many of the classical polynomials, for example, the Laguerre, and [...] those of Jacobi, Legendre".

The Theorem

Theorem

Let F be proper hypergeometric written in the form (16)*. Then there exist $M, N \in \mathbb{N}_0$ and polynomials $a_{r,s} : \mathbb{Z} \to \mathbb{C}$ for $r = 0, \ldots, M$ and $s = 0, \ldots, N$ which are not all identical to* 0*, such that the recurrence* (5) *holds.*

Remark

Furthermore, it can be shown that such a recurrence holds for

$$M = \sum_{s=1}^{U} |b_s| + \sum_{s=1}^{V} |v_s| \quad \text{and} \tag{19}$$

$$N = 1 + \deg(P) + M \cdot \left(-1 + \sum_{s=1}^{U} |a_s| + \sum_{s=1}^{V} |u_s| \right). \tag{20}$$

Sketch of proof

It is possible (and also performed in my summary) to explicitly construct a polynomial $\Delta : \mathbb{R}^2 \to \mathbb{R}$ as well as polynomials $\nu_{s,r} : \mathbb{R}^2 \to \mathbb{R}, \delta_{s,r} : \mathbb{R}^2 \to \mathbb{R}$ for all $s \in \{0, \ldots, N\}$ and $r \in \{0, \ldots, M\}$, such that the recurrence (5) can be written as

$$\sum_{r=0}^{M} \sum_{s=0}^{N} a_{r,s}(n) \underbrace{\nu_{s,r}(n,k) \frac{\Delta}{\delta_{s,r}(n,k)}}_{\text{polynomial in } n \text{ and } k} = 0. \tag{21}$$

All that is left to do is to collect powers of k and set all coefficients of the powers of k to 0. This amounts in a linear equation system. It is thus enough to show that by choosing M, N large enough, we obtain a linear equation system with more unknowns than equations.

Sketch of proof

By construction, there are $(M+1)(N+1)$ unknowns. The number of equations is equal to the number of (distinct) powers of k that appear in the sum (21). It can be shown that this number grows only like a linear function of N and M. This growth is much slower than the growth of $(M+1)(N+1)$ so we are done.

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Example 1

Consider the sum

$$G(n) = \sum_{k=0}^{n} k^3 \binom{n}{k}.$$

Example 1

Using the Mathematica script, more specifically the function `findRecur`, we find that the $F(n,k) = k^3\binom{n}{k}$ satisfy the recurrence (5) with $M = N = 2$ and the coefficients

$$(a_{r,s}(n))_{r,s\in\{0,1,2\}} =$$

$$\begin{pmatrix} \dfrac{\dfrac{\left(-5n^2+11n-6\right)a_{1,1}(n)}{n} - \dfrac{\left(10n^3-33n^2+29n-2\right)a_{0,1}(n)}{(n-1)^2}}{4(n+2)} & a_{0,1}(n) \\ \dfrac{\dfrac{n\left(10n^3-37n^2+37n-4\right)a_{0,1}(n)}{(n-1)^3}+(5n-8)a_{1,1}(n)}{2(n+2)} & a_{1,1}(n) \\ -\dfrac{n\left(10n^2-21n+3\right)a_{0,1}(n)+5(n-1)^3a_{1,1}(n)}{4(n-1)^2(n+2)} & -\dfrac{n\left(n^2-6n+1\right)a_{0,1}(n)+(n-1)^3a_{1,1}(n)}{(n-1)^2(n+2)} \end{pmatrix}$$

Example 1

Here is the last column of the matrix (i.e. $s = 2$)

$$\left. \begin{matrix} 0 \\ -\frac{n^3 a_{0,1}(n)}{(n-1)^3} \\ \frac{\frac{n\left(6n^2+3n-1\right)a_{0,1}(n)}{(n-1)^2}+(n-1)a_{1,1}(n)}{4(n+2)} \end{matrix} \right) \tag{23}$$

Since we have more unknowns than linear equations, the coefficients $a_{0,1}(n)$ and $a_{1,1}(n)$ are freely choosable.

Example 1

For example, I will choose (see also the attached Mathematica script) $a_{0,1}(n) = (n-1)^2$ and $a_{1,1}(n) = n$. This arbitrary choice was made in order to cancel some of the denominators. From Lemma 5, which was implemented as `findRecurForSum` in the Mathematica script, we get the linear recurrence relation

$$4(n-5)n^2 G(n-2) + 2(13-4n)n^2 G(n-1) + \left(3n^3 - 14n^2 + 15n - 2\right) G(n) = 0. \quad (24)$$

Example 1

Considering that $G(0) = 0$ and $G(1) = 1$, we get using algorithm Hyper (described in the talks by Zouhair Ouaggag and Stefan Herytash) that

$$\boxed{G(n) = \sum_{k=0}^{n} k^3 \binom{n}{k} = 2^{n-3} n^2 (3+n).} \tag{25}$$

Example 2

Consider the sum

$$G(n) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} (-2)^{n-k}.$$

Example 2

We find that the $F(n,k) = \binom{n}{k}\binom{2k}{k}(-2)^{n-k}$ satisfy the recurrence (5) with $M = 2, N = 1$ and the coefficients

$$(a_{r,s}(n))_{r,s} = \begin{pmatrix} a_{0,0}(n) & 0 \\ \frac{2(2n-1)a_{0,0}(n)}{n} & -\frac{2(2n-1)a_{0,0}(n)}{n} \\ \frac{4(n-1)a_{0,0}(n)}{n} & -\frac{8(n-1)a_{0,0}(n)}{n} \end{pmatrix}, \qquad (26)$$

where $a_{0,0}(n)$ is again freely choosable.

Example 2

I will choose $a_{0,0}(n) = n$, since this cancels all the denominators in the other coefficients, and by Lemma 5, I obtain the recurrence

$$nG(n) = 4(n-1)G(n-2). \tag{27}$$

Together with $G(0) = 1$ and $G(1) = 0$, it follows that

$$\boxed{G(n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even} \end{cases}.} \tag{28}$$