Zeilberger's Algorithm

Existence of the telescoped recurrence

Alain Schmid

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University of Zurich

1. Setting

1. Setting - Basics

Convention: $0 \in \mathbb{N}$.

We consider a sum of the form

$$f(n) = \sum_k F(n, k),$$

for n, k such that the terms $F(n, k)$ are well-defined and hypergeometric.

1. Setting - Hypergeometric terms

Recall the following definition:

Definition
A term $F(n,k)$ is a **hypergeometric term** *in both arguments, if*

$$\frac{F(n+1,k)}{F(n,k)} \qquad \textit{and} \qquad \frac{F(n,k+1)}{F(n,k)}$$

are both rational functions of n and k.

1. Setting - Goal

Zeilberger's algorithm makes use of the existence of a **telescoped recurrence**, therefore we want to prove that

there exists $J \in \mathbb{N}_{>0}$, polynomials $(a_j)_{j=0}^{J}$ in $\mathbb{C}[n]$, not all zero, and a term $G(n,k)$ such that

$$\sum_{j=0}^{J} a_j(n) F(n+j,k) = G(n,k+1) - G(n,k)$$

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Assuming G has finite support, summing on both sides over k gives

$$\sum_{j=0}^{J} a_j(n) f(n+j) = 0.$$

2. Statement

2. Statement - Precondition

Recall:

Definition
A term $F(n,k)$ is **proper hypergeometric** *if it can be written in the form*

$$F(n,k) = P(n,k)\,\frac{\prod_{i=1}^{q}(a_i n + b_i k + c_i)!}{\prod_{j=1}^{r}(u_i n + v_i k + w_i)!}x^k,$$

where $x \in \mathbb{C}$, $P \in \mathbb{C}[n,k]$, $a_i, b_i, u_j, v_j \in \mathbb{Z}$ for all $i \in \{1,...,r\}, j \in \{1,...,q\}$ and $r, q \in \mathbb{Z}_{\geq 0}$.

2. Statement - Main theorem

Theorem
Let $F(n,k)$ be a proper hypergeometric term. Then there are $J \in \mathbb{N}_{>0}$, polynomials $(a_j)_{j=0}^{J}$ in $\mathbb{C}[n]$, not all zero, and a function $G(n,k)$ such that (whenever $F(n,k) \neq 0$ and all appearing terms of the form $F(n+j,k)$ are well-defined)

$$\sum_{j=0}^{J} a_j(n)\, F(n+j,k) = G(n,k+1) - G(n,k)$$

and $\frac{G(n,k)}{F(n,k)}$ is a rational function.

3. Preparation for the proof

3. Preparation for the proof - Fundamental theorem

Recall the first part of the fundamental theorem:

Theorem
Let $F(n,k)$ be a proper hypergeometric term. Then there exist $I, J \in \mathbb{N}_{>0}$ and polynomials $a_{ij} \in \mathbb{C}[n]$ for $i = 0, ..., I, j = 0, ..., J$, not all zero, such that

$$\sum_{i=0}^{I} \sum_{j=0}^{J} a_{ij}(n) F(n-j, k-i) = 0$$

whenever $F(n,k) \neq 0$ and all appearing terms of the form $F(n-j, k-i)$ are well-defined and non-zero.

3. Preparation for the proof - Shift operators

Definition
If $p(n)$ (resp. $u(k)$) is a term dependent on n (resp. k), then define

$$N(p(n)) := p(n+1) \qquad \text{and} \qquad K(u(k)) := u(k+1).$$

3. Preparation for the proof - Shift operators

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I will use distributive notation, i.e.

$$\left(aN^n K^k + bK^l N^m + c\right)(F(n,k)) := \\ aN^n \left(K^k (F(n,k))\right) + bK^l (N^m (F(n,k))) + cF(n,k)$$

for $a, b, c \in \mathbb{C}, n, k, l, m \in \mathbb{N}$.

3. Preparation for the Proof - Lemma 1

Lemma 1
Let $P \in \mathbb{C}[u, v, w]$ *be a polynomial. Then there exists a polynomial* $Q \in \mathbb{C}[u, v, w]$ *such that*

$$P\left(u, v, w\right) = P\left(u, v, 1\right) + \left(1 - w\right) Q\left(u, v, w\right).$$

3. Preparation for the Proof - Lemma 1 - Proof

Proof
Let $P \in \mathbb{C}[u, v, w]$ be a polynomial. Its Taylor series in $w = 1$ is

$$\sum_{n=0}^{\infty} \frac{\frac{\partial^n P}{\partial w^n}(u, v, 1)}{n!}(w-1)^n,$$

a finite sum (since P is a polynomial). Thus there exists a polynomial $Q \in \mathbb{C}[u, v, w]$ such that

$$P(u, v, w) = P(u, v, 1) + (1 - w) Q(u, v, w),$$

i.e. choose

$$Q(u, v, w) := -\left(\sum_{n=1}^{\infty} \frac{\frac{\partial^n P}{\partial w^n}(u, v, 1)}{n!}(w-1)^{n-1}\right).$$

■

3. Preparation for the Proof - Lemma 2

Lemma 2
Let $Q \in \mathbb{C}[x, y, z]$ *and* $F(n, k)$ *be a hypergeometric term in both arguments. Then* $Q(N, n, K)\, F(n, k)$ *is a rational multiple of* $F(n, k)$.

Lemma 2
Let $Q \in \mathbb{C}[x, y, z]$ and $F(n, k)$ be a hypergeometric term in both arguments. Then $Q(N, n, K) F(n, k)$ is a rational multiple of $F(n, k)$.

Proof
Let $F(n, k)$ be a hypergeometric term in both arguments, fulfilling the conditions above, and $Q \in \mathbb{C}[x, y, z]$. Then $\frac{Q(N,n,K)F(n,k)}{F(n,k)}$ can be written in the form

$$\sum_{(i,j)\in A} a_{ij}(n) \frac{F(n+i, k+j)}{F(n,k)}$$

for some finite $A \subset \mathbb{N}^2$ and $a_{ij} \in \mathbb{C}[n]$ for all $(i, j) \in A$.

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for some finite $A \subset \mathbb{N}^2$ and $a_{ij} \in \mathbb{C}[n]$ for all $(i,j) \in A$.

Note that for each $(i,j) \in A$

$$F(n+i,k+j) = \frac{F(n+i,k+j)}{F(n+i-1,k+j)} F(n+i-1,k+j)$$

and $\frac{F(n+i,k+j)}{F(n+i-1,k+j)}$ is a rational function.

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and $\frac{F(n+i,k+j)}{F(n+i-1,k+j)}$ is a rational function.

Thus (by iterating this procedure) there is some rational function $R(n,k)$ such that

$$F(n+i,k+j) = R(n,k)\, F(n,k+j)\,.$$

3. Preparation for the Proof - Lemma 2 - Proof

Thus (by iterating this procedure) there is some rational function $R(n,k)$ such that

$$F(n+i,k+j) = R(n,k)\,F(n,k+j)\,.$$

With the same argument as before, we find also a rational function $S(n,k)$ such that

$$F(n,k+j) = S(n,k)\,F(n,k)\,.$$

Thus

$$\frac{F(n+i,k+j)}{F(n,k)} = R(n,k)\,S(n,k)\,.$$

■

4. Proof

4. Proof - Strategy

A Take the provided 2-variable recurrence from the fundamental theorem.

B Reorder the terms such that we get a recurrence in the desired form which is independent of k.

C Show (from this form) that the provided function $G(n, k)$ is a rational multiple of $F(n, k)$.

D Prove by contradiction that the found recurrence is nontrivial.

4. Proof - Step A

Let $F(n,k)$ be a proper hypergeometric term. Then, using the first part of the fundamental theorem, there exist $I, J \in \mathbb{N}_{>0}$ and polynomials $a_{ij} \in \mathbb{C}[n]$ for $i \in \{0, ..., I\}, j \in \{0, ..., J\}$, not all zero, such that

$$\sum_{i=0}^{I} \sum_{j=0}^{J} a_{ij}(n) F(n+j, k+i) = 0$$

whenever $F(n,k) \neq 0$ and all of the values $F(n+j, k+i)$ are well-defined.

4. Proof - Step A

Let $F(n,k)$ be a proper hypergeometric term. Then, using the first part of the fundamental theorem, there exist $I, J \in \mathbb{N}_{>0}$ and polynomials $a_{ij} \in \mathbb{C}[n]$ for $i \in \{0, ..., I\}, j \in \{0, ..., J\}$, not all zero, such that

$$\sum_{i=0}^{I} \sum_{j=0}^{J} a_{ij}(n) F(n+j, k+i) = 0 \tag{1}$$

whenever $F(n,k) \neq 0$ and all of the values $F(n+j, k+i)$ are well-defined.

Using the notion of shift operators, (1) can be written in the form

$$P(N, n, K)(F(n,k)) = 0 \tag{2}$$

for some polynomial $P \in \mathbb{C}[u, v, w|$.

4. Proof - Step B

It's possible to find a polynomial $Q \in \mathbb{C}[u, v, w]$ such that

$$P\left(u, v, w\right) = P\left(u, v, 1\right) + \left(1 - w\right) Q\left(u, v, w\right). \tag{3}$$

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It's possible to find a polynomial $Q \in \mathbb{C}[u, v, w]$ such that

$$P\left(u, v, w\right) = P\left(u, v, 1\right) + \left(1 - w\right) Q\left(u, v, w\right).$$

Plugging this into (2) yields

$$P\left(N, n, 1\right)\left(F\left(n, k\right)\right) + \left(1 - K\right)\left(Q\left(N, n, K\right)\left(F\left(n, k\right)\right)\right) = 0$$

which is equivalent to

$$P\left(N, n, 1\right)\left(F\left(n, k\right)\right) = \left(K - 1\right)\left(Q\left(N, n, K\right)\left(F\left(n, k\right)\right)\right). \tag{4}$$

4. Proof - Step B

$$P(N, n, 1)(F(n, k)) = (K - 1)(Q(N, n, K)(F(n, k)))$$

Define

$$G(n, k) := Q(N, n, K)(F(n, k)),$$

then

$$P(N, n, 1)(F(n, k)) = (K - 1)\, G(n, k) = G(n, k + 1) - G(n, k). \quad (5)$$

We have

Lemma 2
Let $Q \in \mathbb{C}[x, y, z]$ and $F(n, k)$ be a hypergeometric term in both arguments. Then $Q(N, n, K) F(n, k)$ is a rational multiple of $F(n, k)$.

and

$$G(n, k) := Q(N, n, K)(F(n, k)).$$

4. Proof - Step D

Note that

$$P\left(N, n, K\right) \not\equiv 0.$$

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$$P(N,n,K) \not\equiv 0.$$

Choose $P(N,n,K)$ with least possible degree in K such that it fulfills this property and

$$P(N,n,K)(F(n,k)) = 0.$$

4. Proof - Step D

Note that

$$P(N, n, K) \not\equiv 0.$$

Choose $P(N, n, K)$ such that it fulfills this property and

$$P(N, n, K)(F(n, k)) = 0.$$

Again, write

$$P(N, n, K) = P(N, n, 1) + (1 - K)\, Q(N, n, K).$$

4. Proof - Step D

Again, write

$$P(N, n, K) = P(N, n, 1) + (1 - K) Q(N, n, K).$$

Assume that

$$P(N, n, 1) \equiv 0$$

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4. Proof - Step D

Again, write

$$P\left(N,n,K\right)=P\left(N,n,1\right)+\left(1-K\right)Q\left(N,n,K\right).$$

Assume that

$$P\left(N,n,1\right)\equiv 0$$

and recall that

$$P\left(N,n,K\right)F\left(n,k\right)=0$$

.

4. Proof - Step D

Again, write

$$P(N, n, K) = P(N, n, 1) + (1 - K)\, Q(N, n, K)\,.$$

Assume that

$$P(N, n, 1) \equiv 0$$

and recall that

$$P(N, n, K)\, F(n, k) = 0.$$

Thus

$$G(n, k+1) - G(n, k) = (K - 1)\,(Q(N, n, K)\,(F(n, k))) = 0.$$

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Thus

$$G(n, k+1) - G(n, k) = (K-1)(Q(N, n, K)(F(n, k))) = 0.$$

Therefore we find a rational function

$$g(n) := \frac{G(n+1, k)}{G(n, k)}.$$

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Thus

$$G(n,k+1) - G(n,k) = (K-1)(Q(N,n,K)(F(n,k))) = 0.$$

Therefore we find a rational function

$$g(n) := \frac{G(n+1,k)}{G(n,k)}.$$

Rearranging,

$$(N - g(n))(G(n,k)) = 0.$$

Using, that g is rational, we find $a, b \in \mathbb{C}[n]$ such that

$$g(n) = \frac{a(n)}{b(n)}$$

and

$$(b(n)N - a(n))\,G(n,k) = 0$$

4. Proof - Step D

What we have:

$$G(n,k) := Q(N,n,K)(F(n,k))$$

$$P(N,n,K) = P(N,n,1) + (1-K)\,Q(N,n,K)$$

$P(N,n,K) \not\equiv 0$ has minimal degree in K such that
$P(N,n,K)(F(n,k)) = 0$

$$P(N,n,1) \equiv 0$$

$$(b(n)\,N - a(n))\,G(n,k) = 0$$

4. Proof - Step D

Case I: $Q \equiv 0$
Immediately from our assumptions

$$P(N, n, K) \equiv P(N, n, 1)$$

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Case I: $Q \equiv 0$
Immediately from our assumptions

$$P\left(N, n, K\right) \equiv P\left(N, n, 1\right),$$

but $P\left(N, n, K\right) \not\equiv 0$ and $P\left(N, n, 1\right) \equiv 0$.

Case II: $Q \not\equiv 0$
As we found,

$$(b(n)N - a(n))(Q(N, n, K)(F(n, k))) = 0$$

is a nontrivial recurrence for $F(n, k)$.

4. Proof - Step D

Case II: $Q \not\equiv 0$
As we found,

$$(b(n)N - a(n))(Q(N,n,K)(F(n,k))) = 0$$

is a nontrivial recurrence for $F(n,k)$. Recalling

$$P(N,n,K) = P(N,n,1) + (1-K)\,Q(N,n,K)$$

we get a contradiction to the minimality of the degree in K of $P(N,n,K)$.

4. Proof - Conclusion

We can conclude that

$P\,(N, n, 1)\,(F\,(n, k)) = (K - 1)\,G\,(n, k)$ is a nontrivial telescoped recurrence for $F\,(n, k)$.

THE END