The WZ phenomenon (Part 1)

Elena Silingardi

14.05.20

Table of Contents

Table of Contents

The WZ method: The Algorithm

1. Suppose we want to show $\sum_k t(n,k) = \text{RHS}(n)$.
2. If $\text{RHS}(n) \not\equiv 0$, look instead at $f(n) := \sum_k F(n,k) = 1$, where $F(n,k) = \frac{t(n,k)}{\text{RHS}(n)}$.
3. Find $R(n,k)$, the certificate for the WZ method (see later, steps 3.1 and 3.2) and define $G(n,k) := R(n,k)F(n,k)$.
4. Check that $D := F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$ (mostly easily verifiable). Summing this equation over all k and noting that the RHS telescopes to 0, we get $\sum_k F(n+1,k) = \sum_k F(n,k)$, i.e. $\sum_k F(n,k)$ doesn't depend on n and is therefore constant.
5. Verify that this constant is indeed 1 by entering an explicit value for n, e.g. check that $\sum_k F(0,k) = 1$.

The WZ method: WZ pair

Definition

A pair $(F(n,k), G(n,k))$ is called a **WZ pair**, if it fulfills

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

The WZ method: Step 3: Find $R(n, k)$

For the crucial Step 3 of the WZ method we use Gosper's algorithm in the following way:

3.1 Let $D(k) := F(n+1, k) - F(n, k)$ and input $D(k)$ into Gosper's algorithm. If Gosper fails, then the WZ method fails as well.

3.2 Otherwise Gosper gives us a function $g(k)$ such that $D(k) = g(k+1) - g(k)$. Of course this g will also contain a parameter n, so rename it to $G(n, k)$. This $G(n, k)$ is the WZ mate for $F(n, k)$. Furthermore $\frac{G(n,k)}{F(n,k)} := R(n, k)$ is a rational function.

The WZ method: Do we have a problem?

As seen in a previous presentation: for every proper hypergeometric term $F(n, k)$ we *always* have a telescoping certification of $\sum_k F(n, k) = \text{const.}$ as

$$\sum_{j=0}^{J} a_j(n) F(n + j, k) = G(n, k + 1) - G(n, k)$$

If we divide by the RHS (if RHS $\neq 0$), then very often this LHS just reduces to $F(n + 1, k) - F(n, k)$.

Table of Contents

The WZ method is valid: Theorem

Theorem

Let (F, G) *be a WZ pair, i.e. s.t.* $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$. *Assume*

(G1) $\forall n \in \mathbb{N} : \lim_{k\to\pm\infty} G(n,k) = 0$

Then $\sum_k F(n,k) = const.\ \forall n \in \mathbb{N}$, *i.e. the certification procedure is valid.*

The WZ method is valid: Proof

Let $\Delta_n h(n) := h(n+1) - h(n)$.
Starting from the WZ equation and summing over $-L \leqslant k \leqslant K$ we get:

$$\sum_{k=-L}^{K} (F(n+1,k) - F(n,k)) = \sum_{k=-L}^{K} (G(n,k+1) - G(n,k))$$

But the summand on the LHS is just $\Delta_n(F(n,k))$ and the RHS telescopes to $G(n,K+1) - G(n,-L)$. Therefore we get:

$$\Delta_n \left(\sum_{k=-L}^{K} F(n,k) \right) = G(n,K+1) - G(n,-L)$$

The WZ method is valid: Proof

Now, taking the limits as $K, L \to +\infty$ we get on the LHS $\Delta_n(\sum_k F(n,k)) = \sum_k F(n+1,k) - \sum_k F(n,k)$ and on the RHS, by (G1), we get 0. So overall:

$$\sum_k F(n+1,k) = \sum_k F(n,k)$$

This means, that $\sum_k F(n,k)$ is independent of n, i.e. it is constant.

Table of Contents

Mathematica session: The WZ module

```
(*WZ::usage=
 "WZ[f,n,k] yields the WZ certificate of f[n,k]. Here the input f is an
 expression, not a function. If verbose=TRUE, R(n,k) and G(n,k) are printed.
   If check=TRUE, 3 checks are done and printed. For that the bounds of the
   supports of F (sF, eF) and G (sG, eG) are needed.
 R(n,k) and G(n,k) are returned as expressions."*)
WZnew[f_, n_, k_, verbose_, check_, sG_, eG_, sF_, eF_] :=
  Module[{df, r, g, WZcheck, TelCheck, ConstCheck},
   df = (f /. {n → n + 1}) - f;  (*D(k)=F(n+1,k)-F(n,k)*)
   (*Input D(k) into Gosper's algorithm --> Output: g(k) s.t. g(k+1)-g(k)=D(k)*)
   g = FactorialSimplify[GosperSum[df, k]];
   r = FactorialSimplify[g / f];  (*R(n,k)=G(n,k)/F(n,k)*)
   If[verbose == TRUE,
    Print["The rational function R(n,k) is ", r];
    Print["The WZ mate G(n,k) is ", g];
   ];
   If [check == TRUE,
    (*Need F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k) for valid WZ pair*)
    WZcheck = FactorialSimplify[(f /. {n → n + 1}) - f - (g /. {k → k + 1}) + g];
    Print["Check that (F,G) is a valid WZ pair (should give 0): ", WZcheck];
    (*Need RHS of WZ equation to telescope to 0. Then Sum(F(n,k)) is independent
     of n, i.e. constant*)
    TelCheck = Sum[(g /. {k → k + 1}) - g, {k, sG, eG}];
    Print["Check that RHS telescopes to 0 (should give 0): ", TelCheck];
    (*Check that this constant is 1 by entering some value for n, e.g. n=1*)
    ConstCheck = Sum[(f /. {n → 1}), {k, sF, eF}];
    Print["Check that Sum(F(n,k))=1 (should give 1):", ConstCheck];
   ];
   {r, g} (*Return R(n,k) and G(n,k)*)
  ];
```

Example: Gauss's ${}_2F_1$ identity

For $b \in \mathbb{Z}_- \cup \{0\}$ or $Re(c-a-b) > 0$ we have:

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} $$

By definition we can rewrite the LHS as:

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{1^k}{k!} $$

$$ = \sum_{k=0}^{+\infty} \frac{(a+k-1)!(b+k-1)!(c-1)!}{(a-1)!(b-1)!(c+k-1)!k!} $$

Example: Gauss's ${}_2F_1$ identity

For $b \in \mathbb{Z}_- \cup \{0\}$ or $Re(c-a-b) > 0$ we have:

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} $$

Now, the LHS we can rewrite as:

$$ \sum_{k=0}^{+\infty} \frac{(a+k-1)!(b+k-1)!(c-1)!}{(a-1)!(b-1)!(c+k-1)!k!} $$

Then, dividing this new LHS by the RHS and setting $a = n$ to get an identity of the form $\sum_k F(n,k) = 1$ we get:

$$ \sum_{k=0}^{+\infty} \frac{(n+k-1)!(b+k-1)!(c-1)!\Gamma(c-n)\Gamma(c-b)}{(n-1)!(b-1)!(c+k-1)!k!\Gamma(c-n-b)\Gamma(c)} $$

$$ = \sum_{k=-1}^{+\infty} \frac{(n+k)!(b+k)!(c-1)!\Gamma(c-n)\Gamma(c-b)}{(n-1)!(b-1)!(c+k)!(k+1)!\Gamma(c-n-b)\Gamma(c)} \stackrel{!}{=} 1 $$

Example: Gauss's ${}_2F_1$ identity

$$\sum_{k=-1}^{+\infty} \frac{(n+k)!(b+k)!(c-1)!\Gamma(c-n)\Gamma(c-b)}{(n-1)!(b-1)!(c+k)!(k+1)!\Gamma(c-n-b)\Gamma(c)} = 1$$

This summand we can now enter in Mathematica:

```
f := ((n+k)!(b+k)!(c-1)!Gamma[c-n]Gamma[c-b])/
((n-1)!(b-1)!(c+k)!(k+1)!Gamma[c-n-b]Gamma[c]);
WZnew[f, n, k, TRUE, FALSE, , , ,];
```

And we get out:

The rational function R(n,k) is $-\frac{(1+k)(c+k)}{(-1+c-n)\ n}$

The WZ mate G(n,k) is $-\frac{(-1-b+c)!(b+k)!(-2+c-n)!(k+n)!}{(-1+b)!k!(-1+c+k)!(-1-b+c-n)!n!}$

Example: Gauss's ${}_2F_1$ identity

We want to check that things worked correctly.

```
WZnew[f, n, k, FALSE, TRUE, -1, Infinity, -1,
Infinity];
```

And indeed we get out:

```
Check that (F,G) is a valid WZ pair (should give
0):  0
Check that RHS telescopes to 0 (should give 0):
0
Check that Sum(F(n,k))=1 (should give 1):1
```

Table of Contents

The companion identity: Theorem

Theorem

Let (F, G) be a WZ pair, i.e. s.t. $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$. Assume

(F1) *$\forall k$ in the support of $F(n,k)$: $f_k := \lim_{n\to+\infty} F(n,k)$ exists and is finite*

(G2) $\lim_{L\to+\infty} \sum_{n=0}^{+\infty} G(n,-L) = 0$

Then $\sum_{n=0}^{+\infty} G(n,k) = \sum_{j=-\infty}^{k-1}(f_j - F(0,j))$, which we call the ***companion identity****.*

The companion identity: Proof

Let $\Delta_n h(n) := h(n+1) - h(n)$.
Starting from the WZ equation and summing over $0 \leqslant n \leqslant N$ we get:

$$\sum_{n=0}^{N} (F(n+1,k) - F(n,k)) = \sum_{n=0}^{N} (G(n,k+1) - G(n,k))$$

But the LHS telscopes to $F(N+1,k) - F(0,k)$ and the summand on the RHS is just $\Delta_k(G(n,k))$. Therefore we get:

$$F(N+1,k) - F(0,k) = \Delta_k \left(\sum_{n=0}^{N} G(n,k) \right)$$

The companion identity: Proof

Now, taking the limits as $N \to +\infty$ we get on the LHS, by (F1) $f_k - F(0,k)$ and on the RHS $\Delta_k(\sum_{n=0}^{+\infty} G(n,k))$. If we then sum over $-L \leqslant j \leqslant k-1$ we get overall:

$$\sum_{j=-L}^{k-1} (f_j - F(0,j)) = \sum_{j=-L}^{k-1} \Delta_j \left(\sum_{n=0}^{+\infty} G(n,j) \right)$$

But switching the summation signs on the RHS we get $\sum_{n=0}^{+\infty} \sum_{j=-L}^{k-1} (G(n,j+1) - G(n,j)) =$ $\sum_{n=0}^{+\infty} (G(n,k) - G(n,-L))$. Then taking the limits as $L \to +\infty$ we get on the LHS $\sum_{j=-\infty}^{k-1} (f_j - F(0,j))$ and on the RHS, by (G2), we get $\sum_{n=0}^{+\infty} G(n,k)$. So overall:

$$\sum_{j=-\infty}^{k-1} (f_j - F(0,j)) = \sum_{n=0}^{+\infty} G(n,k)$$

The companion identity: General version

Adapting the proof slightly (summing over $l \leqslant n \leqslant N$ in the beginning, instead of $0 \leqslant n \leqslant N$), we also have the more general companion identity

$$\sum_{n=l}^{+\infty} G(n,k) = \sum_{j=-\infty}^{k-1} (f_j - F(l,j))$$

The companion identity: Example 1

Consider the identity $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$. By dividing through the RHS we get $F(n,k) = \binom{n}{k}^2 / \binom{2n}{n}$.

First, we enter this in Mathematica again to find $R(n,k)$ and $G(n,k)$:

$$R(n,k) = \frac{-3n+2k-1}{2(2n+1)}, \;\; G(n,k) = \frac{(-3n+2k-1)\binom{n}{k}^2}{2(2n+1)\binom{2n}{n}}$$

The companion identity: Example 1

Using Stirling's formula we get $(F1)$ for $F(n,k) = \binom{n}{k}^2 / \binom{2n}{n}$:

$$f_k := \lim_{n\to+\infty} F(n,k) = \dots = 0$$

$$(G2): \lim_{L\to+\infty} \sum_{n=0}^{+\infty} G(n,-L) = \lim_{L\to+\infty} \sum_{n=0}^{+\infty} \frac{-3n-2L-1}{2(2n+1)} \frac{\binom{n}{-L}^2}{\binom{2n}{n}} = 0$$

$$\text{Note that } \quad F(0,k) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{else} \end{cases}$$

Therefore we get as companion identity

$$\left(\sum_{n=0}^{+\infty} G(n,k) = \sum_{j=-\infty}^{k-1} (f_j - F(0,j)) \right):$$

$$\sum_{n=0}^{+\infty} \frac{-3n+2k-1}{2(2n+1)} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = -1 \iff \sum_{n=0}^{+\infty} \frac{3n-2k+1}{2n+1} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = 2$$

The companion identity: Example 1

$$\sum_{n=0}^{+\infty} \frac{3n-2k+1}{2(2n+1)} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = 1$$

We can in fact check this new identity (note that we switch the roles of n and k).

```
f := ((3k-2n+1) Binomial[k, n]^ 2)/(2(2k+1)
Binomial[2k, k]);
WZnew[f, n, k, TRUE, FALSE, , , ,];
```

And we get:

The rational function R(n,k) is $-\frac{2(1+2k)(k-n)^{\wedge}\ 2}{(1+3k-2n)(1+n)^{\wedge}\ 2}$

The WZ mate G(n,k) is $-\frac{(k-n)^{\wedge}\ 2\ \text{Binomial}[k,n]]^{\wedge}\ 2}{(1+n)^{\wedge}\ 2\ \text{Binomial}[2k,k]}$

The companion identity: Example 1

Then to check it:

```
WZnew[f, n, k, FALSE, TRUE, n, Infinity, 1,
Infinity];
```

And after some computation time we are rewarded with:

```
Check that (F,G) is a valid WZ pair (should give
0):  0
Check that RHS telescopes to 0 (should give 0):
0
Check that Sum(F(n,k))=1 (should give 1):1
```

The companion identity: Example 2

Consider the following identity:

$$\sum_{k=1}^{+\infty} \frac{(n-i)!(n-j)!(i-1)!(j-1)!}{(n-1)!(k-1)!(n-i-j+k)!(i-k)!(j-k)!} = 1$$

To have well-defined factorials, we get the constraints:

$$n \geqslant \max\{i, j, 1\}, \ 1 \leqslant k \leqslant \min\{i, j\}, \ n+k \geqslant i+j \qquad (1)$$

Enter $F(n,k)$ in Mathematica to find $R(n,k)$ and $G(n,k)$:

```
f := ((n-i)!(n-j)!(i-1)!(j-1)!)/
((n-1)!(k-1)!(n-i-j+k)!(i-k)!(j-k)!);
WZnew[f, n, k, TRUE, FALSE, , , ,];
```

And we get out:

The rational function R(n,k) is $\frac{-1+k}{n}$

The WZ mate G(n,k) is $\frac{(-1+i)!(-1+j)!(-i+n)!(-j+n)!}{(i-k)!(j-k)!(-2+k)!n!(-i-j+k+n)!}$

The companion identity: Example 2

Then to check it:

```
WZnew[f, n, k, FALSE, TRUE, 1, Infinity, 1,
Infinity];
```

And we are indeed rewarded with:

```
Check that (F,G) is a valid WZ pair (should give
0):  0
Check that RHS telescopes to 0 (should give 0):
0
Check that Sum(F(n,k))=1 (should give 1):1
```

The companion identity: Example 2

Using Stirling's formula we get $(F1)$ for

$$F(n,k) = \frac{(n-i)!(n-j)!(i-1)!(j-1)!}{(n-1)!(k-1)!(n-i-j+k)!(i-k)!(j-k)!}$$

$$\text{as } f_k := \lim_{n\to+\infty} F(n,k) = \ldots = \begin{cases} 1, & \text{if } k=1 \\ 0, & \text{if } k \geqslant 2 \end{cases}$$

(G2): $\lim_{L\to+\infty} \sum_{n=0}^{+\infty} G(n,-L)$

$$= \lim_{L\to+\infty} \frac{(i-1)!(j-1)!}{(-L-2)!(i+L)!(j+L)!} \sum_{n=0}^{+\infty} \frac{(n-i)!(n-j)!}{n!(n-i-j-L)!} = 0$$

The companion identity: Example 2

$$F(j,k) = \frac{(j-i)!(i-1)!}{(k-1)!(k-i)!(i-k)!(j-k)!} = \binom{j-i}{j-k}\binom{i-1}{i-k}$$

From constraint $n + k \geqslant i + j$, for $n = j$, we get $k \geqslant i$. From second binomial coefficient we need $k \leqslant i$. So in total $k = i$.

Therefore we get as companion identity (for $l = j$):

$$\left(\sum_{n=j}^{+\infty} G(n,k) = \sum_{k'=-\infty}^{k-1} (f_{k'} - F(j,k')) \right)$$

$$\frac{(i-1)!(j-1)!}{(k-2)!(i-k)!(j-k)!} \sum_{n=j}^{+\infty} \frac{(n-i)!(n-j)!}{n!(n-i-j+k)!} = 1 - \sum_{k'=-\infty}^{k-1} \binom{j-i}{j-k'}\binom{i-1}{i-k'}$$

The companion identity: Example 2

$$\frac{(i-1)!(j-1)!}{(k-2)!(i-k)!(j-k)!}\sum_{n=j}^{+\infty}\frac{(n-i)!(n-j)!}{n!(n-i-j+k)!} = 1 - \sum_{k'=-\infty}^{k-1}\binom{j-i}{j-k'}\binom{i-1}{i-k'}$$

For RHS: need $0 \leqslant j-k' \leqslant j-i$ and $0 \leqslant i-k' \leqslant i-1$, i.e. $j \geqslant k' \geqslant i$ and $i \geqslant k' \geqslant 1$, i.e. $k' = i$. But the sum only goes up to $k-1 = i-1$. So the sum in the RHS is 0. Overall:

$$\sum_{n=j}^{+\infty}\frac{(n-i)!(n-j)!}{n!(n-i-j+k)!} = \frac{(k-2)!(i-k)!(j-k)!}{(i-1)!(j-1)!}$$

If we now write $n-i-j+k = r$ in the sum on the LHS and set $i = c-a$, $j = c-b$ and $k = c-a-b+1$, we get

$$\sum_{r=0}^{+\infty}\frac{(r+a-1)!(r+b-1)!}{r!(r+c-1)!} = \frac{(c-a-b-1)!(b-1)!(a-1)!}{(c-a-1)!(c-b-1)!}$$

which is exactly Gauss's ${}_2F_1$ identity.