The WZ phenomenon

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Table of Contents

Table of Contents

Dual Identities: Introduction

- Recall that $(F(n,k), G(n,k))$ is called a **WZ pair**, if it fulfills

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

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- Idea: transform WZ pairs $(F(n,k), G(n,k))$ to new WZ pairs $(\tilde{F}(n,k), \tilde{G}(n,k))$

- Multiply the hypergeometric terms $F(n,k)$ and $G(n,k)$ by a certain periodic meromorphic function $P(n,k)$.

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- If $P(n,k)$ is a periodic function of n and k of period 1, then

$$\tilde{F}(n+1,k) - \tilde{F}(n,k) - \tilde{G}(n,k+1) + \tilde{G}(n,k)$$

$$= F(n+1,k)P(n+1,k) - F(n,k)P(n,k) - G(n,k+1)P(n,k+1) + G(n,k)P(n,k)$$

$$= F(n+1,k)P(n,k) - F(n,k)P(n,k) - G(n,k+1)P(n,k) + G(n,k)P(n,k) = 0$$

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- New identity is called **dual identity** of $\sum_k F(n,k) = const.$
- It is called dual, because the mapping $(F(n,k), G(n,k)) \longrightarrow (\tilde{F}(n,k), \tilde{G}(n,k))$ is an involution.

Table of Contents

Dual Identities: The Method

- Consider the hypergeometric term

$$F(n,k) = x^k \rho(n,k) \frac{\prod_i (a_i n + b_i k + c_i)!}{\prod_i (u_i n + u_i k + w_i)!}$$

with ρ a rational function of n, k.

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  - Take a factor $(an + bk + c)!$ that appears in the numerator (or denominator) of F. Remove it.

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- To obtain some F_{new} do the following:
  - Take a factor $(an + bk + c)!$ that appears in the numerator (or denominator) of F. Remove it.
  - Place the factor $(-1 - an - bk - c)!$ in the denominator (numerator).
  - Multiply everything by $(-1)^{an+bk}$.

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- Operation can be carried out on any factorial factor in numerator or denominator of F and G.
- Can be performed repeatedly on different factorial factors, while preserving the WZ pair relationship.

Table of Contents

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We define

$$F(n,k) := \frac{\binom{a}{k}\binom{n}{k}}{\binom{n+a}{a}} = \frac{(a!)^2(n!)^2}{(k!)^2(a-k)!(n-k)!(n+a)!}$$

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Vandermonde identity $\Rightarrow \sum_k F(n,k) = 1$.

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Out= The WZ mate G(n,k) is $\frac{\texttt{k\^{}2 Binomial[a,k]Binomial[n,k]}}{\texttt{(-1+k-n)(1+a+n)Binomial[n+a,a]}}$

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The result is

$$F_{new}(n,k) = \frac{(-1)^{-k}(a!)^2(n!)^2(-1-a+k)!}{(k!)^2(n-k)!(n+a)!}.$$

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$$\tilde{F}(n,k) = \frac{(-1)^{-k+n}(n!)^2(-1-a+k)!(-1-a-n)!}{(k!)^2(n-k)!(-1-a)!^2}.$$

Dual Identities: Example

Applying exactly the same operations choosing the same factors to

$$G(n,k) = \frac{(a!)^2(n!)^2k^2}{(k!)^2(a-k)!(n-k)!(n+a)!(-1+k-n)(1+a+n)},$$

then we get WZ mate of $\tilde{F}$, namely

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Are $\tilde{F}$ and $\tilde{G}$ really a WZ pair?

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In:= FactorialSimplify[F_tilde[n + 1, k] -
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Out= 0
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Dual Identities: Example

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$$\tilde{G}(n,k) = \frac{(-1)^{-k+n}(n!)^2(-1-a+k)!(-1-a-n)!k^2}{(k!)^2(n-k)!(-1-a)!^2(-1+k-n)(1+a+n)}$$
$$= \frac{(-1)^{n+k}\binom{n}{k}\binom{k-1-a}{-1-a}k^2}{\binom{-1-a}{n}(1+k-n)(1+a+n)}$$

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$$\sum_k \tilde{F}(n,k) = 1 \Leftrightarrow \sum_k (-1)^{n+k} \binom{n}{k} \binom{k-1-a}{-1-a} = \binom{-1-a}{n}.$$

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- (Check, if $(\tilde{F}(n,k), \tilde{G}(n,k))$ really WZ pair)
- Check (G1)
- Write new identity $\sum_k (-1)^{n+k}\binom{n}{k}\binom{k-1-a}{-1-a} = \binom{-1-a}{n}$.

Table of Contents

Dual Identities: Proof

Prop 2.1

The pair $(\tilde{F}, \tilde{G})$, obtained by dualizing a WZ pair (F, G) is again a WZ pair.

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$$(an + bk + c)! = -\frac{\pi}{sin(\pi(an + bk + c))(-1 - an - bk - c)!}.$$

The claim follows from the reflection formula

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}.$$

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Substituting $(an + bk + c + 1)$ for z, we get

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Substituting $(an + bk + c + 1)$ for z, we get

$$\begin{aligned}
(an+bk+c)! &= \Gamma(z) \\
&= \frac{\pi}{\sin(\pi z)\Gamma(1-z)} \\
&= \frac{\pi}{\sin(\pi(an+bk+c+1))(-an-bk-c-1)!} \\
&= -\frac{\pi}{\sin(\pi(an+bk+c))(-an-bk-c-1)!}.
\end{aligned}$$

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We therefore multiply $F(n, k)$ by a function

$$P(n,k) := \frac{(-1)^{an+bk+1}\sin(\pi(an+bk+c))}{\pi}$$

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$$\begin{aligned} P(n+1,k) &= \frac{(-1)^{a(n+1)+bk+1}\sin(\pi(a(n+1)+bk+c))}{\pi} \\ &= \frac{(-1)^{an+bk+1+a}(-1)^{a}\sin(\pi(an+bk+c))}{\pi} \\ &= P(n,k) \end{aligned}$$

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\begin{aligned}
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&= \frac{(-1)^{an+bk+1+a}(-1)^{a}\sin(\pi(an+bk+c))}{\pi} \\
&= P(n, k)
\end{aligned}
$$

P(n,k) periodic $\Rightarrow$

$(\tilde{F}(n, k), \tilde{G}(n, k)) = (F(n, k) \cdot P(n, k), G(n, k) \cdot P(n, k))$ is again a WZ pair.

Dual Identities: Proof

- Problem: Reflection formula does not hold for integers. Replace $(an + bk + c)!$ by an expression like $\Gamma(an + bk + c + 1 + \epsilon)$, carry out the calculations and then take the limit.

Dual Identities: Proof

- Why is $(F(n,k), G(n,k)) \longrightarrow (\tilde{F}(n,k), \tilde{G}(n,k))$ an involution?

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$$F(n,k) \xrightarrow[\frac{(-1)^{an+bk+1}}{(an+bk+c)!(-1-an-bk-c)!}]{\text{dualizing first time}} F_{new1}(n,k)$$

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$$F_{new1}(n,k) \xrightarrow[(-1)^{-an-bk-1}(-1-an-bk-c)!(-1+1+an+bk+c)]{\text{dualizing second time}} F_{new2}(n,k)$$

In summary, when applying the method twice, we multiply F by

$$\frac{(-1)^{an+bk}(-1)^{-an-bk}(-1-an-bk-c)!(-1+1+an+bk+c)!}{(an+bk+c)!(-1-an-bk-c)!},$$

which is 1.

Table of Contents

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- Consider again WZ pair $(F(n,k), G(n,k))$.

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  Recall that $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$.

- Assume $\sum_k F(n,k) = 1$ and support of F properly contains the interval $[0,n]$.

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- Assume $\sum_k F(n,k) = 1$ and support of F properly contains the interval $[0,n]$.

- Object of interest:

$$h(n) := \sum_{k=0}^{n} F(n,k).$$

The Definite Sum Made Indefinite: The Method

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- Derive identity:
  - Sum $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$ over $k \leq n$,

The Definite Sum Made Indefinite: The Method

- Derive identity:
  - Sum $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$ over $k \leq n$,
  - obtain

    $$h(n+1) - F(n+1,n+1) - h(n) = G(n,n+1) - G(n,0).$$

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  - $\Leftrightarrow$

    $$\sum_{k=0}^{n} F(n,k) = F(0,0) + \sum_{j=1}^{n} (F(j,j) + G(j-1,j) - G(j-1,0)).$$

$(\star)$

Table of Contents

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$$G(n,k) = \frac{(-1)^{k-n} k^2 (10n^2 - 6kn + 17n + k^2 - 5k + 7) \binom{2n}{k}^2}{2(2n-k+2)^2 (2n-k+1)^2 \binom{2n}{n}}$$

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Therefore $(\star)$ reads

$$\sum_{k=0}^{n} \frac{(-1)^{k-n}\binom{2n}{k}^2}{\binom{2n}{n}} = 1 + \sum_{j=1}^{n} \frac{\binom{2j}{j}(3j-1)}{4(2j-1)}$$

End and Prospect

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- But there are many more possibilities to transform WZ pairs to new WZ pairs, for example:

$$(F(n,k), G(n,k)) \longrightarrow (G(-k-1,-n), F(-k,-n-1))$$