A combinatorial approach of representations of symmetric groups : application to Kerov's polynomials

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Plan

1. Introduction
   - Free cumulants and Kerov's polynomials
   - Combinatorial formula for characters

Plan

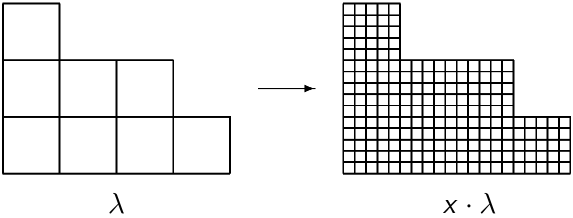
Free cumulants

$$\begin{array}{rcl} \text{Young diagram } \lambda & \rightarrow & \text{Transition measure} \\ & \rightarrow & \text{Free cumulants } (R_i(\lambda))_{i \geq 2} \end{array}$$

Proposition (Biane)

Homogeneous $R_i(x \cdot \lambda) = x^i R_i(\lambda)$

Asymptotics $\chi^{x \cdot \lambda}(1 \dots k) \sim_{x \to \infty} R_{k+1}(\lambda) |x \cdot \lambda|^{-k/2}$



Kerov's polynomials

If $\mu \in S(k) \subset S(n)$ and $\lambda \vdash n$, let

$$\Sigma_\mu^\lambda = n(n-1)\dots(n-k+1)\frac{\chi^\lambda(\mu)}{\chi^\lambda(Id_n)}$$

where χ^λ is the character of the irreducible representation indexed by λ.

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Theorem (Existence of Kerov's polynomials, Kerov, Biane)

*Let $k \geq 1$, there exists a **universal** polynomial K_k such that :*

$$\Sigma_{(1\dots k)}^\lambda = K_k(R_2(\lambda), \dots, R_{k+1}(\lambda))$$

It does not depend on the diagram λ!

Description of the coefficients

Asymptotics property of free cumulants implies :

Proposition

$$K_k = R_{k+1} + \textit{ lower degree terms}$$

Moreover :

- It has integer coefficients.

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Moreover :

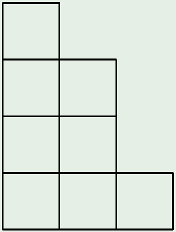
- It has integer coefficients.
- We will sketch the proof of their positivity thanks to a combinatorial description using permutations in $S(k)$.

Irreducible representations of symmetric groups

- They are indexed by partitions $\lambda \vdash n$, or equivalently by Young diagrams.

Example

- $\lambda_1 = 3; \lambda_2 = \lambda_3 = 2;$ $\lambda_4 = 1; \lambda_5 = \ldots = 0,$



Irreducible representations of symmetric groups

- They are indexed by partitions $\lambda \vdash n$, or equivalently by Young diagrams.
- Other notation : $\lambda = \mathbf{p} \times \mathbf{q}$.

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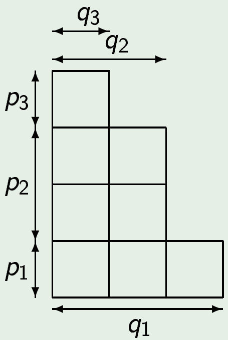
- $\lambda_1 = 3; \lambda_2 = \lambda_3 = 2;$
  $\lambda_4 = 1; \lambda_5 = \ldots = 0,$
- $\lambda = (1,2,1) \times (3,2,1)$



Map of a pair of permutations

pair of permutations $\mapsto$ bicolored edge-labeled map

Example

$$\tau = (14)(325), \overline{\tau} = (13)(254)$$

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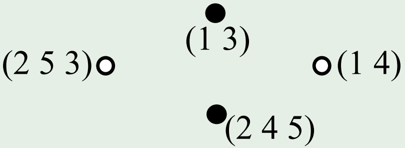

white vertices	$\leftrightarrow$	cycles of τ
black vertices	$\leftrightarrow$	cycles of $\overline{\tau}$

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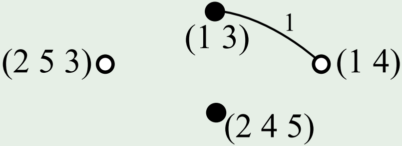

The edge labeled 1 links the two vertices corresponding to cycles containing 1.

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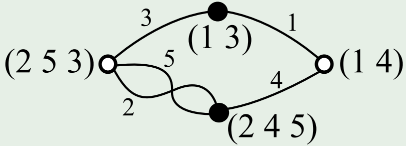

Same thing for the integers between 2 and k. The cyclic order at each vertex is given by the corresponding cycle.

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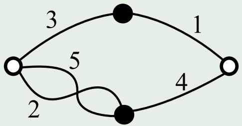

We can recover the pair of permutations from the map.

Power series associated to a bicolored map

A colouring of the white vertices of M is :

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$$\psi : \begin{array}{rcl} V_b(M) & \to & \mathbb{N}^\star \\ b & \mapsto & \max\limits_{w \text{ neighbour of } b} \varphi(w) \end{array}$$

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Let define the power series in indeterminates $\mathbf{p}$ and $\mathbf{q}$:

$$N(M) = \sum_{\substack{\varphi \text{ colouring of} \\ \text{the white vertices}}} \left(\prod_{w \in V_w(M)} p_{\varphi(w)} \prod_{b \in V_b(M)} q_{\psi(b)} \right)$$

Combinatorial formulas for character values and cumulants

Theorem (Stanley, Féray, Śniady)

With these notations, the character values is given by :

$$\Sigma_{\mu}^{\mathbf{p}\times\mathbf{q}} = \sum_{\substack{\tau,\overline{\tau}\in S(k)\\ \tau\cdot\overline{\tau}=\mu}} (-1)^{|C(\overline{\tau})|} N(M^{\tau,\overline{\tau}})(\mathbf{p},\mathbf{q})$$

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From asymptotic property of cumulants, we have :

$$R_{k+1}(\mathbf{p}\times\mathbf{q}) = \sum_{\substack{\tau,\overline{\tau}\in S(k)\\ \tau\cdot\overline{\tau}=(1\dots k)\\ |C(\tau)|+|C(\overline{\tau})|=k+1}} (-1)^{|C(\overline{\tau})|} N(M^{\tau,\overline{\tau}})(\mathbf{p},\mathbf{q})$$

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The factorisation appearing in the second equation are in bijection with $NC(k)$ (**non-crossing partitions** of $[k]$). They are exactly the pair of permutations whose map is a **planar tree**.

Idea

Recall that, as power series in **p** and **q** :

$$\Sigma_k = K_k(R_2, \dots, R_{k+1})$$

Replace R_i by their combinatoric expression and expand, we obtain something of the kind:

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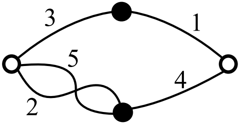
We will write each summand under the form :

$$N^{\tau,\overline{\tau}} = \sum \pm \text{ series associated to forests}$$

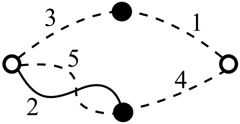
Elementary transformation

Description on our favorite example



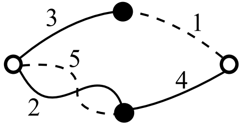
Elementary transformation

We choose a loop (here dotted)



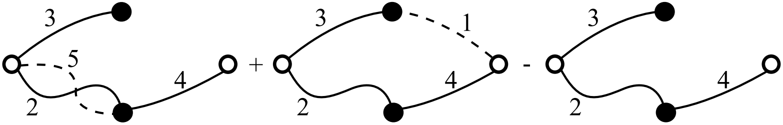
Elementary transformation

Call erasable one edge over two of this loop



Elementary transformation

Let $T_{\vec{L}}(M)$ be the formal expression :



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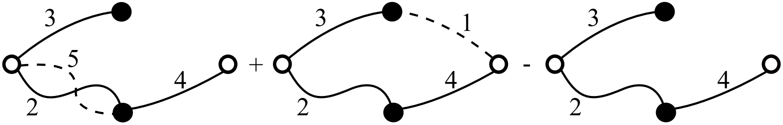

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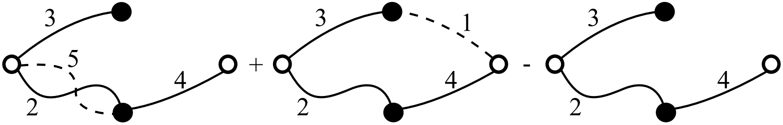

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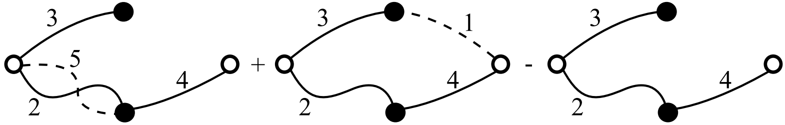

Proposition

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Different choices of loops and erasable edges $\longrightarrow$ **Maybe different** ways to write $N(M)$ as $\sum N(\text{forest})$

Restriction of choices

To obtain a particular decomposition, we will specify some choices :

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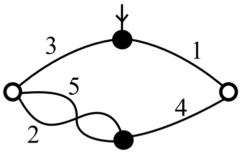
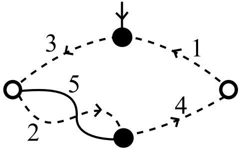
1. Add an external half-edge of black extremity to connected components which do not have one (after the edge of smallest label) and draw it on top on the map.



Restriction of choices

To obtain a particular decomposition, we will specify some choices :

1. Add an external half-edge to connected components which do not have one.
2. In any connected component, choose an admissible oriented loop : a loop going through $\star$ oriented from left to right if there is some.



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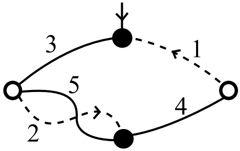
To obtain a particular decomposition, we will specify some choices :

1. Add an external half-edge to connected components which do not have one.
2. In any connected component, choose an admissible oriented loop.
3. Select the edges which are oriented **from** their **white** extremity **to** their **black** extremity in $\vec{L}$.



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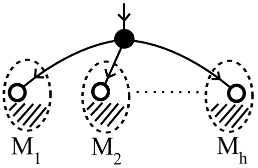
3. Select the edges which are oriented from white to black in $\vec{L}$.



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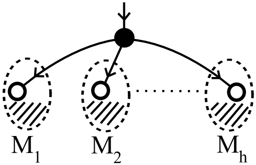

If we iterate transformations with such choices of erasable edges, we obtain an algebraic sum of forests, whose associated polynomial is equal to $N(M)$.

Invariance of the result

There is still some choices to do, but :

Proposition

If we follow the rules above, we always obtain the ***same*** *sum of forests which we will denote $D(M)$.*

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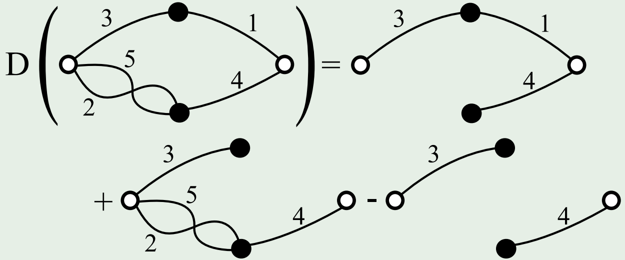
Example



Properties of our decomposition

As we iterate $N-$invariant transformations :

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Thanks to our choice of loops, one has :

Proposition

The sign of the coefficient of M' in $(-1)^{\#\text{ c.c. of } M} D(M)$ is $(-1)^{\#\text{ c.c. of } M'}$

Back to Kerov's polynomials

Recall :

$$\Sigma_k = \sum_{\substack{\tau,\overline{\tau} \in S(k) \\ \tau \cdot \overline{\tau} = (1 \dots k)}} (-1)^{|C(\overline{\tau})|} N(M^{\tau,\overline{\tau}})$$

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Replace each term $N(M)$ by $N(D(M))$, we have something like :

$$\Sigma_k = \sum \pm N(forests)$$

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To understand Kerov's polynomial we have to put terms together and make appear free cumulants.

Order on the symmetric group

Definition

$$|\sigma| \overset{\text{def}}{:=} \min \Big\{ h \text{ s.t. } \exists \textit{ transpositions } \tau_1, \dots, \tau_h \\ \text{with } \sigma = \tau_1 \cdot \tau_2 \cdot \dots \cdot \tau_h \Big\}$$

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Proposition

If $\sigma \leq \sigma'$ and $\sigma^{-1}\sigma' = c_1 \cdot \ldots \cdot c_t$ (decomposition in cycles of disjoint supports),

$$[\sigma; \sigma'] \simeq [id_k; \sigma^{-1}\sigma'] \simeq \prod [e; c_i] \simeq \prod NC(|c_i| + 1)$$

Intervals and cumulants

Let ϕ be an isomorphism :

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If $\tau \in [\sigma;\sigma']$, denote :

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If we choose well ϕ, $N_\phi(\tau)$ appears in $N(D(M))$. So intervals are a good tool to make appear products of free cumulants in Σ_k.

Main theorem

With an appropriate family of isomorphisms ϕ, we prove :

Theorem

If $\mu \in S(k)$, let

$$\Sigma'_\mu \overset{def}{:=} \sum_{\substack{\tau,\overline{\tau}\in S(k)\\ \tau\cdot\overline{\tau}=\mu \\ <\tau,\overline{\tau}> \text{ trans.}}} (-1)^{|C(\overline{\tau})|+|C(\mu)|-1} N(M^{\tau,\overline{\tau}})$$

*then there exists a polynomial with **non-negative** integer coefficients such that :*

$$\Sigma'_\mu = K'_\mu(R_2, \dots, R_k)$$

The case $|C(\mu)| = 1$ is the result we claimed for classical Kerov's polynomial.

Computation of coefficients

Proposition

The coefficient of monomial $\prod_{i=1}^{t} R_{j_i+1}$ *in* K'_μ *is the coefficient of the disjoint union of* t *trees with one black and respectively* $j_1, \dots, j_t$ *white vertices in*

$$\sum_{\substack{\tau,\overline{\tau}\in S(k) \\ \tau\overline{\tau}=\sigma, <\tau,\overline{\tau}> \text{trans.} \\ |C(\overline{\tau})|=t}} D(M^{\tau,\overline{\tau}}).$$

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Corollary

The coefficient of the linear monomial R_d *in* K_k *is the number of cycles* $\sigma \in S(k)$ *such that* $\sigma^{-1}(12\dots k)$ *has* $d-1$ *cycles.*

Computation of coefficients

Corollary

The coefficient of the linear monomial R_d in K_k is the number of cycles $\sigma \in S(k)$ such that $\sigma^{-1}(12\dots k)$ has $d-1$ cycles.

Proof.

If $|C(\overline{\tau})| = 1$, the map $M = M^{\tau,\overline{\tau}}$ has one black vertex, so $D(M)$ is a tree with one black vertex and as many white vertices as M. □

End

Thank you!