Quasi-symmetric functions as polynomials on Young diagrams

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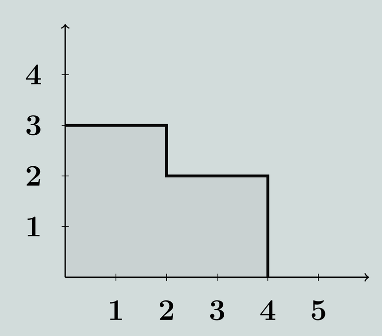
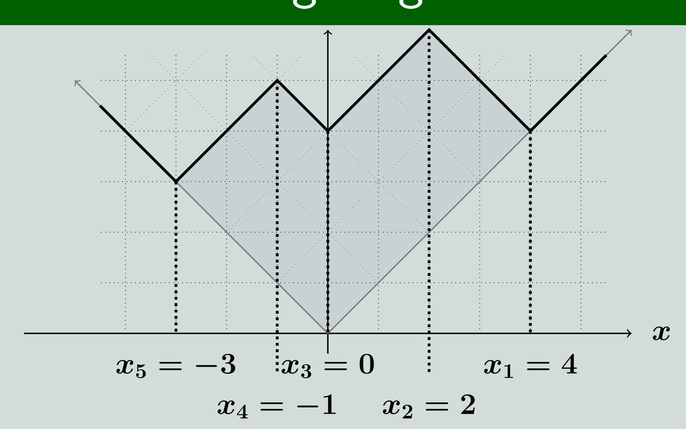
Goal: characterize polynomials in infinitely many variables s.t.

$$f(x_1, x_2, \dots)|_{x_i=x_{i+1}} = f(x_1, \dots, x_{i-1}, x_{i+2}, \dots) \qquad (1)$$

The main problem

Motivation : smooth functions on Young diagrams





(Kerov interlacing coordinates)

One can allow two coordinates to be equal but in that case, there are (infinitely) many encodings of the same diagram. A function f on the x_i satisfies (1) iff it takes the same value on all these encodings.

Motivating example: the *normalized irreducible character value of the symmetric group* $\lambda \mapsto \hat{\chi}^\lambda(\pi)$ on a *fixed permutation* $\pi \in S_k$ is (up to a multiplicative constant) such a *smooth function on Young diagrams*.

A virtual alphabet

Notation: M_I monomial basis of $QSym$; S_i set of generators of Sym.
Let $\mathbb{X}$ be the virtual alphabet defined by

$$\sigma_1(\mathbb{X}A) = \sum_I M_I(\mathbb{X}) S^I(A) = \overrightarrow{\prod_{i\geq 1}} \sigma_{x_i}(A)^{(-1)^i}, \qquad (2)$$

where $\sigma_{x_i}(A) = 1 + x_i S_1 + x_i^2 S_2 + x_i^3 S_3 + \dots$
For example,

$$M_{(k)}(\mathbb{X}) = -x_1^k + x_2^k - x_3^k + x_4^k - \dots$$
$$M_{(k,\ell)}(\mathbb{X}) = \sum_i x_{2i+1}^{k+\ell} + \sum_{i<j} (-1)^{i+j} x_i^k x_j^\ell$$

The $M_I(\mathbb{X})$ indeed remain the same when one puts $x_{k+1} = x_k$.

First result: solution to (1) in the commutative framework

A function f satisfies the functional equation (1) if and only if $f \in QSym(\mathbb{X})$.

Links with other results

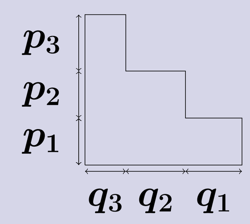
- Natural generalization of the algebra of *polynomial functions on Young diagrams* considered by Kerov and Olshanski (which corresponds to $Sym(\mathbb{X})$).
- Also extends a result of Stembridge.
  - Stembridge's problem: find solutions of (1) which are in addition symmetric in the odd-indexed variables and separately in the even-indexed variables.
  - Stembridge's solution: symmetric functions evaluated on $\mathbb{X}$.

Multirectangular coordinates

The multirectangular coordinates are related to interlacing coordinates by the following changes of variables: for all $i \leq m$,

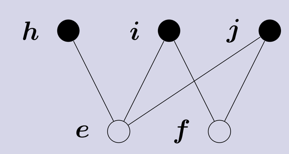
$$\begin{cases} p_i = x_{2i-1} - x_{2i} \\ q_i = x_{2i} - x_{2i+1} \end{cases}$$

so that the p_i and q_i read on a Young diagram as



The N_G functions

Consider a bipartite graph G:



Then define

$$F_G(x_1, x_2, \dots) = \sum_{\substack{e \leq h,i,j \\ f \leq i,j}} x_e\, x_f\, x_h\, x_i\, x_j;$$

$$N_G\begin{pmatrix} p_1\ p_2 \dots \\ q_1\ q_2 \dots \end{pmatrix} = \sum_{\substack{e \leq h,i,j \\ f \leq i,j}} p_e\, p_f\, q_h\, q_i\, q_j.$$

The F_G functions have been studied by Stanley, Gessel (gf of P-*partitions*).

Second result: duplicating alphabets

When expressed on the x_i, the functions N_G satisfy (1).
Furthermore, for any bipartite graph G with vertex set $V = V_\circ \sqcup V_\bullet$,

$$N_G\begin{pmatrix} p_1\ p_2 \dots \\ q_1\ q_2 \dots \end{pmatrix} = (-1)^{|V_\circ|} \Phi_{x \to p,q}(F_G(\mathbb{X})),$$

where $\Phi_{x\to p,q}$ consists in expressing a function of the x_i on the p_i and q_i.

→ The expression of N_G (on *two alphabets*) is entirely encoded in the expression of F_G (on *only one alphabet*).

On characters and N_G functions

- Normalized characters write combinatorially in terms of N_G.
- The N_G span linearly $QSym(\mathbb{X})$, but are not in $Sym(\mathbb{X})$.
- Hope: our result could help to manipulate expressions in terms of N_G...

Noncommutative generalization: Solve (1) when f (written P) is now a noncommutative polynomial

$WQSym$ and a virtual alphabet

Notation: P_u monomial basis of $WQSym$, indexed by *packed words*.

We define $P_u(\mathbb{A})$ as :

- if u is nondecreasing, $P_u(\mathbb{A})$ is the noncommutative analogue of $M_{\mathrm{eval}(u)}(\mathbb{X})$, where x_k is replaced by a_k and all letters in any monomial of $P_u(\mathbb{A})$ are in nondecreasing order.

  ex: $$P_{112}(\mathbb{A}) = \sum_i a_{2i+1}^3 + \sum_{i<j} (-1)^{i+j} a_i a_i a_j;$$

- For other u, define P_u by an action of the symmetric group.

  ex: $$P_{121}(\mathbb{A}) = \sum_i a_{2i+1}^3 + \sum_{i<j} (-1)^{i+j} a_i a_j a_i$$

  is obtained by swapping the second and third letter in each monomial of $P_{112}(\mathbb{A})$.

Third result: solution to (1) in the noncommutative framework

A noncommutative polynomial P satisfies (1) if and only if $P \in WQSym(\mathbb{A})$.

On virtual alphabets for $WQSym$

- There is no formula analog to (2) to define $P_u(\mathbb{A})$;
- It is therefore surprising that such a simple definition of $P_u(\mathbb{A})$ works.
- the functional equation (1) helps proving that $F \to F(\mathbb{A})$ defines an algebra morhpism!

Fourth result: a combinatorial by-product

Let $\mathbb{K}$ be the smallest two-sided ideal of $WQSym$ containing P_1 and whose homogeneous components $\mathbb{K}_n$ are stable by the action of S_n. Then the dimension of $\mathbb{K}_n$ is the number of set-compositions of $\{1, \dots, n\}$ with an odd number of parts.