Universal limits of substitution-closed classes

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(joint work with F. Bassino, M. Bouvel, L. Gerin, M. Maazoun, A. Pierrot; see preprint [2])

Introduction: The general problem we are interested in is the following: consider a permutation class $\mathcal{C}$, and take, for each $n \geq 1$, a random permutation $\boldsymbol{\sigma_n}$, uniformly among permutations of $\mathcal{C}$ of size n, i.e. $\boldsymbol{\sigma_n} \in_u (\mathcal{C} \cap S_n)$. We aim at describing the asymptotic properties of $\boldsymbol{\sigma_n}$ when n tends to infinity.

This is a very vast question, and the kind of answer we might expect depends very much on the class and on the asymptotic properties we are considering. In this work:

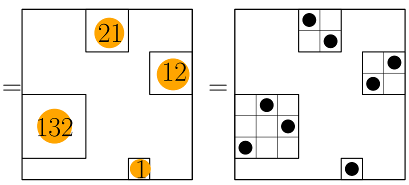
(1) we do not focus on a single permutation class (as in many previous works), but consider a family of classes defined by a closure property, namely classes closed by *substitution*. These classes (there are uncountably many of them) are solvable models, giving a starting point for the asymptotic analysis of random elements.

(2) rather than considering a specific parameter (fixed points, probability that $\boldsymbol{\sigma_n}(i_n) = j_n$ for given i_n and j_n, number of occurrences of patterns) as often done in the literature, we study the scaling limit of $\boldsymbol{\sigma_n}$, in the sense of *permutons*.

Informally, our main result is that, under a quite general analytic assumption, a uniform random permutation $\boldsymbol{\sigma_n}$ in a substitution-closed class converges towards a universal limit, which is a random permuton constructed from the Brownian excursion.

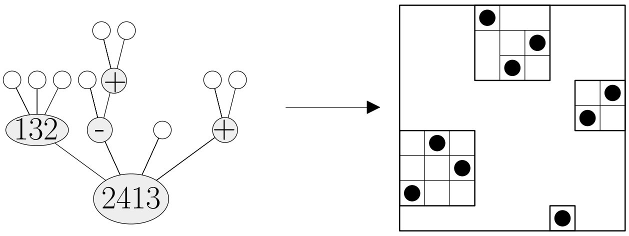
Substitution-closed classes: In the permutation literature, a permutation class is a set of permutations defined by the avoidance of a given set B of patterns (possibly infinite), or equivalently a set stable by taking patterns.

The substitution operation consists in replacing each dot of the diagram of a permutation by the diagram of another permutation. This is better understood on a picture, rather than with a formal definition.

$2413[132, 21, 1, 12] =$ 

$= 24387156$

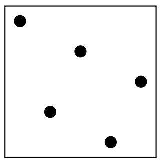
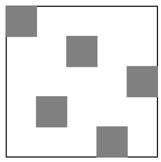
All permutations can be obtained by iterating substitutions operations, starting from "indecomposable elements", which are called simple permutations in this context [1]. This allows to represent permutations as *substitution trees*, whose internal nodes are labelled by simple permutations. Again, we prefer to show an example rather than to give a formal statement.



$= 243978156$

For a substitution-closed class $\mathcal{C}$, a permutation is in $\mathcal{C}$ if and only if all simple permutations in its substitution tree are in $\mathcal{C}$. This allows to relate in a simple way the generating series $C(z)$ of the class to that $S(z)$ of simple permutation in the class.

Permutons: To study limits of combinatorial objects, we need to embed them in some natural way in a (Polish) metric space. For permutations (which we want to identify with their diagrams), it is natural to associate a measure on the unit square $[0,1]^2$. On an example of size $n = 5$:

$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

(Each gray square on the right has a total weight $1/n$, i.e. density n.) The limiting objects are then some probability measures on $[0,1]^2$ (with an additional property, namely having uniform marginals), called *permutons*. This notion was first considered by Hoppen and coauthors, in [3], in analogy with graphons, aka dense graph limits considered by Borgs and co-authors..

We then say that a sequence of (random) permutations converge if the associated (random) measures converge (in distribution) for the weak topology. The following criterion makes this convergence more concrete.

Proposition 1 (BBFGMP, building on a result of Hoppen et al. in the deterministic case)**.** *Let $\boldsymbol{\sigma_n}$ be a sequence of random permutations. Then the following assertions are equivalent:*

(1) *There exists a random permuton $\boldsymbol{\mu}$ such that $\boldsymbol{\sigma_n}$ converges to $\boldsymbol{\mu}$;*
(2) *For each $k \geq 2$, and each pattern τ of size k, the following sequence has a limit:*

$$\mathbb{E}\left(\frac{\#\text{occurrences}(\tau, \boldsymbol{\sigma_n})}{\binom{n}{k}} \right).$$

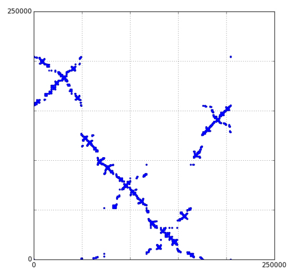
This turns a statement of convergence of random measures into convergence of numbers! Moreover, in the case we are interested in, these expectations can be studied by combinatorial means (counting permutations in the class with a marked occurrence of a given pattern).

Our main result:

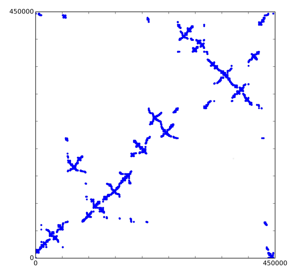
Theorem 1. *Let $\mathcal{C}$ be a substitution-closed class and take $\boldsymbol{\sigma_n} \in_u (\mathcal{C} \cap S_n)$ (for each $n \geq 1$). Assume $S'(R_S) > \frac{2}{(1+R_S)^2} - 1$, where S is the OGF of simple permutations in $\mathcal{C}$ and R_S its radius of convergence. Then $\boldsymbol{\sigma_n}$ tends to the biased Brownian separable permuton $\boldsymbol{\mu}^{(p)}$ for some p in $(0,1)$.*

The limit only depends on the class $\mathcal{C}$ through a parameter p that can be computed from some associated generating series. We do not present the construction of the limit object here, but it can be obtained from a Brownian excursion or from the Brownian continuum random tree [4]. (which is the limit of the substitution tree of $\boldsymbol{\sigma_n}$). We also have some results in the so-called stable and condensation regimes.

Final comments: Most substitution-closed classes we have considered fulfill our analytic assumption, with the notable exception of Av(2413). The proof of Theorem 1 uses the above convergence criterion and analytic combinatorics. Here are some pictures of large permutations in substitution-closed classes, given by their set $\mathcal{S}$ of simple permutations.



Simulation of $\boldsymbol{\sigma}_n$ with $\mathcal{S} = \emptyset$ (separable permutations)



Simulation of $\boldsymbol{\sigma}_n$ with $\mathcal{S} = \{2413,\ 3142,\ 24153\}$

References

[1] M. H. Albert, M. D. Atkinson. Simple permutations and pattern restricted permutations. *Discrete Mathematics*, 300(1):1–15, 2005.

[2] F. Bassino, M. Bouvel, V. Féray, L. Gerin, M. Maazoun, A. Pierrot. Universal limits of substitution-closed permutation classes, `arxiv:1706.08333` (2017).

[3] C. Hoppen, Y. Kohayakawa, C. G. Moreira, B. Rath, R. M. Sampaio. Limits of permutation sequences. *Journal of Combinatorial Theory, Series B*, 103(1): 93–113, 2013.

[4] M. Maazoun, On the Brownian separable permuton. Preprint arXiv:1711.08986.