Brownian limits of large permutations in classes

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Introduction

Permutation classes: set of permutations defining by avoidance of some substructures, called patterns.

Studied thoroughly from enumerative and algorithmic perspective since the 90's.

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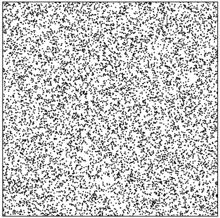
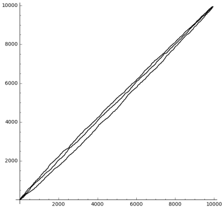
Emerging question: what are the asymptotic properties of a uniform random permutation in a given class?

$\longrightarrow$ this talk: emphasize connections with walks in cones and Brownian limits.

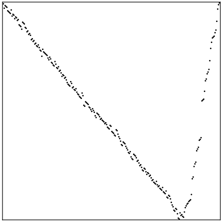
A few uniform random permutations in classes



no constraints



Av(4321) (© Slivken)



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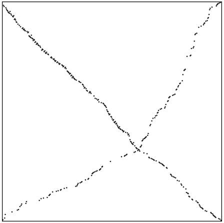

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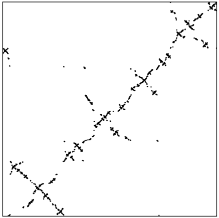
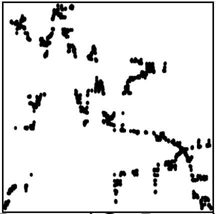


separable



Baxter (© Borga)

(These are diagrams of permutations; a dot at $(i, \sigma(i))$ for each $i \geq 1$.)

Outline of the talk

1. Permutations avoiding monotone patterns and Dyson Brownian bridge (after Hoffman, Rizzolo and Slivken).

2. A universal Brownian limiting object for permutation classes (joint works with Bassino, Bouvel, Gerin, Maazoun and Pierrot and Borga, Bouvel and Stufler).

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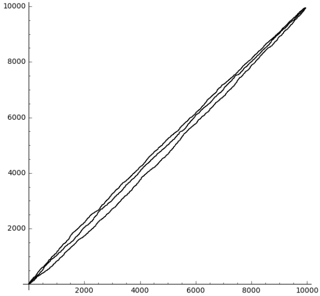
Permutations without decreasing subsequence of size $d+1$...

Simulation for $d = 3$ (©Hoffman, Rizzolo, Slivken)



Permutations without decreasing subsequence of size $d+1\dots$

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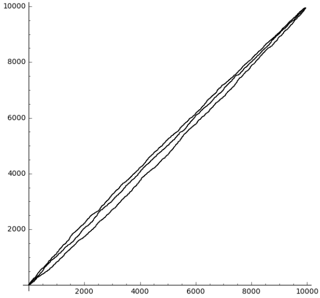

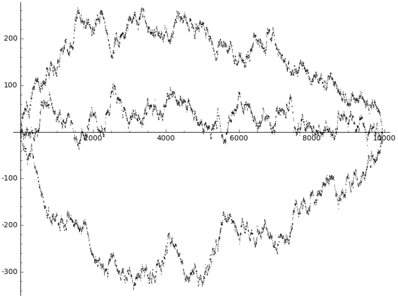

Second picture: exceedance process $(\sigma(i) - i)$.

and d-dimensional Dyson Brownian bridge

For σ in $\mathrm{Av}(d+1\dots1)$, we fix arbitrarily a partition into d increasing sequences (always exists, but not unique).

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For σ in $\mathrm{Av}(d+1\dots1)$, we fix arbitrarily a partition into d increasing sequences (always exists, but not unique).

For $\ell \le d$, we let $(u_i^\ell, \sigma(u_i^\ell))_i$ be the ℓ-th increasing sequence and set

$$s_\sigma^\ell(i) = \left(\frac{u_i^\ell}{n+1}, \frac{\sigma(u_i^\ell) - u_i^\ell)}{\sqrt{n}} \right).$$

Finally, let $\hat{s}_\sigma^\ell$ be the linear interpolation of s_σ^ℓ.

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Theorem (Hoffman, Rizzolo, Slivken, in preparation)

When σ is taken uniformly at random among permutations of size n without decreasing subsequence of size $d+1$, we have

$$(\hat{s}_\sigma^1, \dots, \hat{s}_\sigma^d) \xrightarrow{(d)} \Lambda,$$

where Λ is a traceless d dimensional Dyson Brownian bridge.

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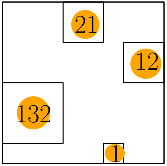
Substitution in permutations

Definition

Let θ be a permutation of size d and $\pi^{(1)}, \dots, \pi^{(d)}$ be permutations. The diagram of the permutation $\theta[\pi^{(1)}, \dots, \pi^{(d)}]$ is obtained by replacing the i-th dot in the diagram of θ with the diagram of $\pi^{(i)}$ (for each i).

$2413[132, 21, 1, 12] =$



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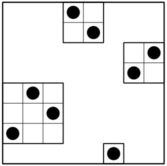


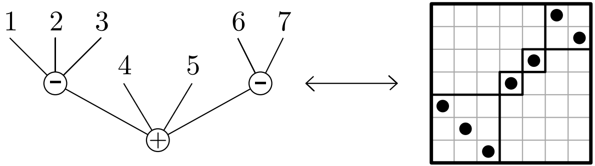
$= 24387156$

Separable permutations and signed Schröder trees

Definition

The class of separable permutations is the smallest set of permutations (of all sizes) containing 1, 12 and 21 and stable by substitution.

Every separable permutations is encoded uniquely by a signed Schröder tree (no unary vertices) with alternating signs.



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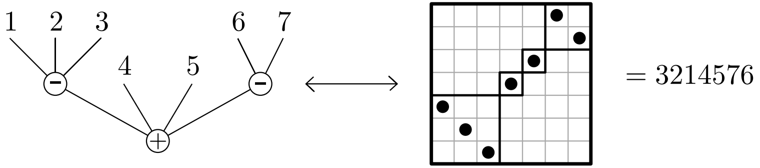
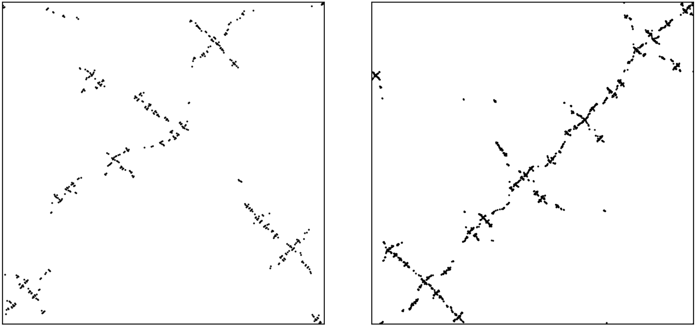


Observation

(i, j) is an inversion in σ if and only if the leaves i and j have a joint labeled with $\ominus$.

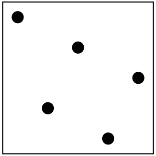
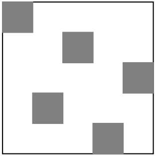
Simulations



Two large uniform random separable permutation

Which notion of convergence? Permutons. . .

A permutation π can be encoded as a probability measure μ_π on $[0,1]^2$.

$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

In μ_π, each small square has weight $1/n$ (i.e. density n).

We have a natural notion of limit for such objects: the weak convergence.
This defines a nice compact Polish space.

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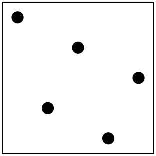
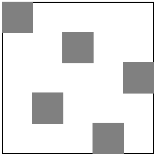
$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

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Note: the projection on μ_π on each axis is the Lebesgue measure on $[0,1]$ (in other words, μ_π has uniform marginals).
$\rightarrow$ potential limits also have uniform marginals.

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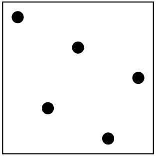
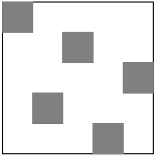
$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

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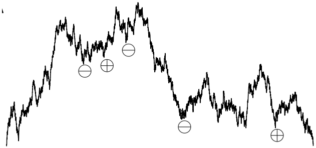
Definition

A permuton is a probability measure on $[0,1]^2$ with uniform marginals.

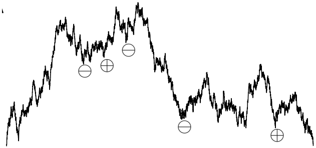
Construction of the limiting object (Maazoun, 2018)



(e, S)

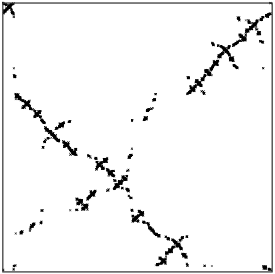
- e is a Brownian excursion and S : $\mathrm{LocalMin}(e) \to \{\oplus, \ominus\}$ is a independent assignment of balanced random signs to local minima of e.

Construction of the limiting object (Maazoun, 2018)



(e, S)

$\mapsto \sigma \mapsto$



$\boldsymbol{\mu} = (x, \sigma(x))_\star(\mathrm{Leb}([0,1])$

- $\sigma : [0,1] \to [0,1]$ is the unique Lebesgue preserving function s.t. (x, y) is an inversion if and only if the sign of $\min_{[x,y]} e$ is $\ominus$.
- The Brownian separable permuton is the "graph of the function σ".

Limits of separable permutations

Theorem (Bassino-Bouvel-F.-Gerin-Pierrot, 2016)

The permuton associated with a uniform random separable permutation of size n converges in distribution to the Brownian separable permuton.

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Later works with Maazoun/Borga and Stufler: the same holds for uniform permutations in

- some classes stable by the substitution operation;
- some classes finitely generated for the substitution operation.

(with appropriate analytic conditions.)

The Brownian separable permuton is in some sense universal.

Proof (1/3): finite dimensional distribution

As for many objects, convergence of permutons is equivalent to convergence of "finite dimensional distribution" and tightness.

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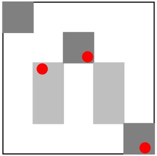
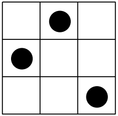
As for many objects, convergence of permutons is equivalent to convergence of "finite dimensional distribution" and tightness.

- The space is compact $\rightarrow$ tightness is automatic.
- What are the finite dimensional distribution of a permuton μ? Take k points i.i.d. according to μ and look at the corresponding permutation:

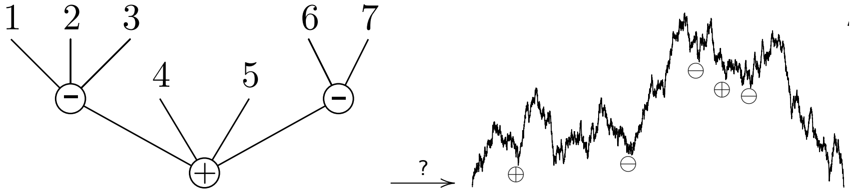


$\mapsto$

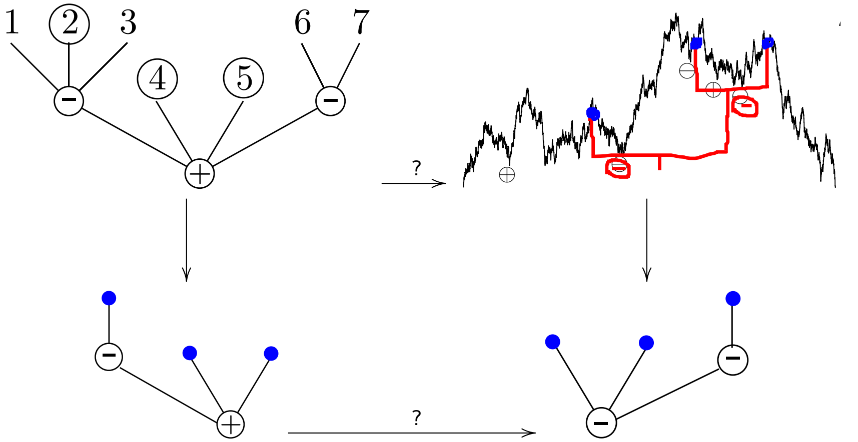


$= 231.$

Proof (2/3): translating into trees



Proof (2/3): translating into trees



We compare finite dimensional distributions. The extracted tree from a Brownian excursion is known to be a *uniform binary tree* (here with independent balanced signs).

Proof (3/3): convergence of extracted signed subtrees

- Without signs: the convergence follows from standard random tree theory (in particular from results of Kortchemski, 2012, Douglas–Rizzolo, 2013).
- For the signs: adhoc exchangeability argument or local limit result for length of branches in random trees.

Transition

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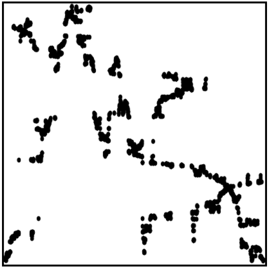
Baxter permutations

Baxter permutations are the permutations avoiding (vincular) patterns $3\underline{14}2$ and $2\underline{41}3$. They are in bijection with families of maps, tilings, pairs of trees.



Question (work in progress by Borga and Maazoun)

What is the limiting permuton?

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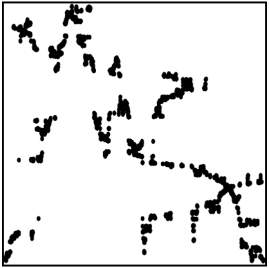
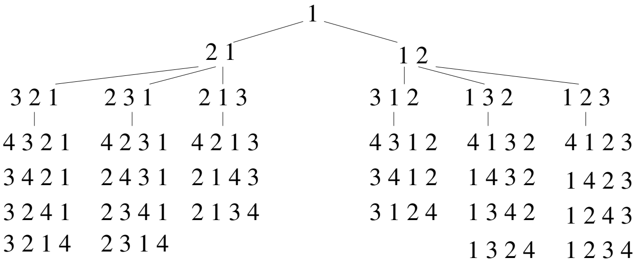


Question (work in progress by Borga and Maazoun)

What is the limiting permuton?

$\rightarrow$ there is an underlying 2D random walks in the positive quadrant. . .

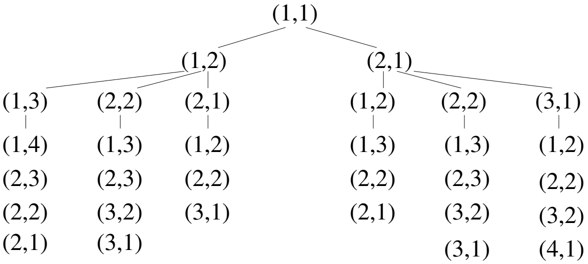
Baxter generating tree



Generating tree of Baxter permutations obtained by adding a new maximal element at each step.

©Guerrini

Baxter generating tree

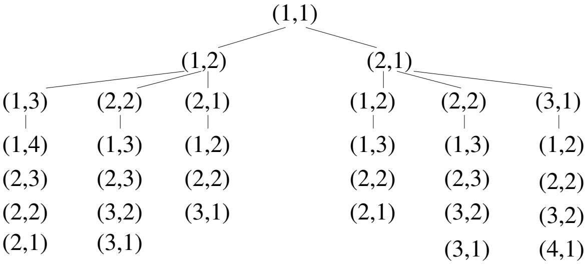


Replace σ by $(\#\text{LRmax}(\sigma), \#\text{RLmax}(\sigma))$.

The resulting tree has a simple rewriting (or offspring) rule:

$$(h,k) \mapsto \begin{array}{l} (1,k+1),(2,k+1),\dots,(h,k+1), \\ (h+1,1),(h+1,2),\dots,(h+1,k). \end{array}$$

Baxter generating tree

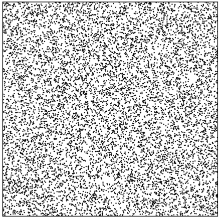


A (random) Baxter permutation is a (random) sequence of labels.

Proposition (Borga, Maazoun, '19)

Taking a uniform random Baxter permutation of size n, the associated random sequence has the distribution of a random walk with i.i.d. steps, conditioned to stay in the positive quadrant and . . .

Thank you for your attention



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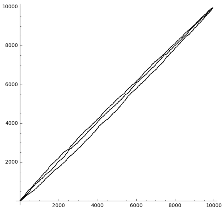

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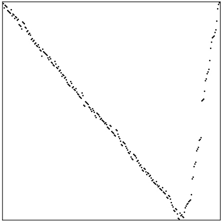

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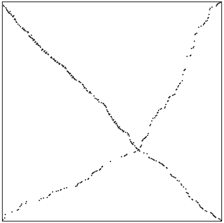

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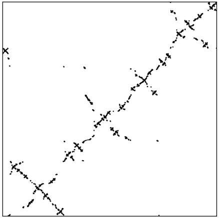
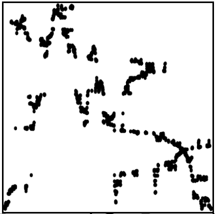


separable



Baxter (© Borga)