Shifted symmetric functions II: expansions in multi-rectangular coordinates

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Séminaire Lotharingien de Combinatoire
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Second lecture

Yesterday, we have seen

- Two nice bases of the shifted symmetric function ring: shifted Schur functions $s^\star_\mu$ and normalized characters Ch_μ.

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- Two nice bases of the shifted symmetric function ring: shifted Schur functions $s^\star_\mu$ and normalized characters Ch_μ.

Today:

- several sets of coordinates of Young diagrams on which we can write shifted symmetric functions (writing in terms of the λ_i is not necessarily the best thing!);
- investigate expansions in one of these sets, multirectangular coordinates.

First part

Equivalent descriptions
of the shifted symmetric function ring

A generating function point of view (Ivanov, Kerov, Olshanski, 2003)

With a Young diagram λ, we associate the function

$$\Phi(\lambda; z) = \prod_{i \geq 1} \frac{z + i}{z - \lambda_i + i}.$$

Around $z = \infty$,

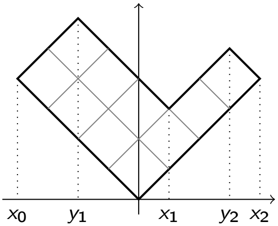
$$\log\big(\Phi(\lambda; z)\big) = \sum_{k \geq 1} \tfrac{1}{k} p_k^\star(\lambda)\, z^{-k},$$

where $p_k^\star(\lambda) = \sum_{i \geq 1} \big[(\lambda_i - i)^N - (-i)^N\big]$.

Proposition

The shifted symmetric ring $\Lambda^\star$ is algebraically generated by the coefficients of the expansion of $\Phi(\lambda; z)$ at $z = \infty$.

Kerov's interlacing coordinates



Alternative description of Young diagrams

x-coordinates of lower corners

$$x_0 = -4,\ x_1 = 1,\ x_2 = 4$$

x-coordinates of higher corners

$$y_1 = -2,\ y_2 = 3$$

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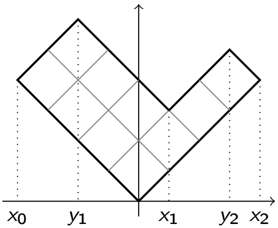

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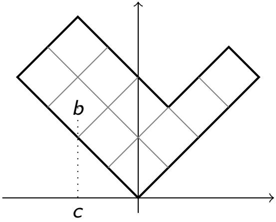
Proposition (IKO '03)

$$\frac{\Phi(\lambda; z-1)}{\Phi(\lambda; z)} = \frac{z \prod_{i=1}^{m} (z - y_i)}{\prod_{i=0}^{m} (z - x_i)}.$$

As a consequence, $\Lambda^\star$ is algebraically generated by the coefficients of the expansion of the RHS at $z = \infty$ (or of $\log(RHS)$).

Proof: $\frac{\Phi(\lambda;z-1)}{\Phi(\lambda;z)} = \frac{z}{z-\lambda_1} \prod_{i \geq 1} \frac{z-\lambda_i+i}{z-\lambda_{i+1}+i}$. Only factors corresponding to corners do not cancel out.

Contents



Content c is the x-coordinate of the box center.

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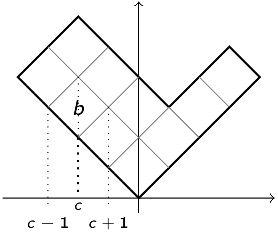

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Note that $c-1$ (resp c, $c+1$, c) are the x-coordinate of the box left (resp. top, right, bottom) corners of the box.

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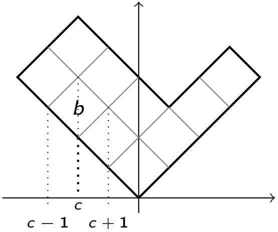

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Claim:

$$\prod_{\square\in\lambda} \frac{(z-c(\square))^2}{(z-c(\square)-1)(z-c(\square)+1)} = \frac{z\prod_{i=1}^m (z-y_i)}{\prod_{i=0}^m (z-x_i)} = \frac{\Phi(\lambda; z-1)}{\Phi(\lambda; z)}.$$

Indeed, in LHS, only factors corresponding to corners of the diagram do not cancel out.

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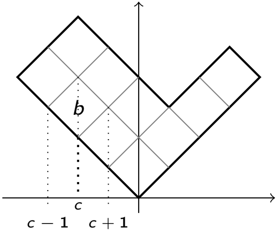

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Indeed, in LHS, only factors corresponding to corners of the diagram do not cancel out. We deduce from this (exercise!)

Proposition

$\Lambda^\star$ is the set of symmetric functions in $\mathcal{C}_\lambda = (c(\square))_{\square\in\lambda}$ with coefficients that are polynomials in $|\lambda|$.

Character on cycles

There is a efficient way to compute $\mathrm{Ch}_{(k)}$ using the function Φ.

Proposition (Frobenius, '00)

$$\mathrm{Ch}_{(k)}(\lambda) = -\frac{1}{k}\,[z^{-1}]\,(z \downharpoonleft k)\frac{\Phi(\lambda; z)}{\Phi(\lambda; z-k)}.$$

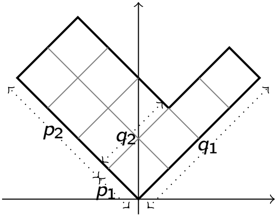
- Generalized by Rattan and Śniady ('06) to several cycles.

Transition

Multirectangular coordinates –
expansion of normalized characters

Multirectangular coordinates (Stanley, 2003)

We see partitions as obtained by piling up rectangles of size $p_i \times q_i$.



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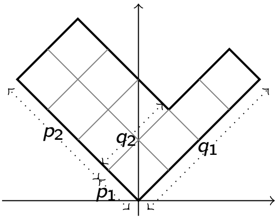

Connection to Kerov's interlacing coordinates

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\begin{aligned}
x_1 &= q_1, & y_1 &= q_1 - p_1 \\
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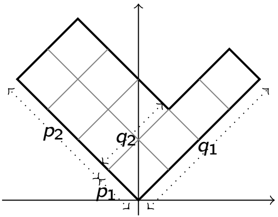

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Why expression in multirectangular coordinates?

- Contains expression in parts (by setting $p_i = 1$), but is more convenient when taking transpose or dilatation of Young diagrams.
- It turns out that these expressions have nice positivity/combinatorial properties!

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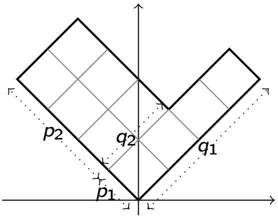

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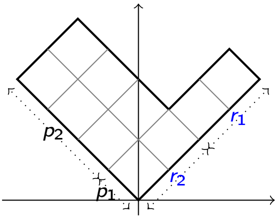

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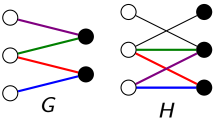
Embeddings of bipartite graphs



Definition: A bipartite graph embedding $G \to H$

maps edges of G to edges of H respecting incidence relations: i.e. edges sharing a black (resp. white) extremity in G are mapped to edges sharing a black (resp. white) extremity in H.

Notation

$$N_G(H) := \#\{\text{embeddings of } G \text{ in } H\}$$

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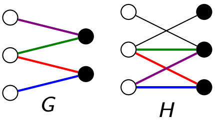
Embeddings of bipartite graphs



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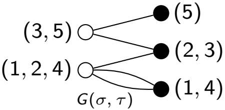
Note: an embedding also maps black (resp. white) vertices of G to black (resp. white) vertices of H

Two families of bipartite graphs associated ...

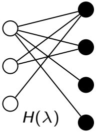
with pairs of permutations



$\sigma = (1,2,4)\,(3,5),\ \tau = (1,4)\,(2,3)\,(5)$

Vertices share as many edges as the size of their intersection.

with partitions



$\lambda = (4, 2, 1)$

The i-th white vertex is connected to the λ_i first black vertices.

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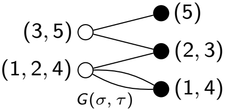

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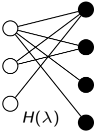

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Short notation: $N_{\sigma,\tau}(\lambda) := N_{G(\sigma,\tau)}\big(H(\lambda)\big)$; $\quad \widetilde{N_{\sigma,\tau}}(\lambda) := \widetilde{N_{G(\sigma,\tau)}}\big(H(\lambda)\big)$.

Lemma

$N_{\sigma,\tau}(\lambda)$ *is a polynomial with nonnegative coefficients in multirectangular coordinates. It has degree* $|C(\sigma)|$ *in* $\boldsymbol{p}$ *and degree* $|C(\tau)|$ *in* $\boldsymbol{q}$ *(or* $\boldsymbol{r}$*).*

A formula for normalized characters

Theorem (Stanley, 2006, F., Śniady, 2007)

Let π be a permutation in S_k of type μ

$$\mathrm{Ch}_\mu = \sum_{\substack{\sigma,\tau \\ \sigma\ \tau=\pi}} (-1)^\tau N_{\sigma,\tau} = \sum_{\substack{\sigma,\tau \\ \sigma\ \tau=\pi}} (-1)^\tau \widetilde{N_{\sigma,\tau}}.$$

$(-1)^\tau =$ sign of $\tau = (-1)^k (-1)^{|C(\tau)|}$.

Note: second equality is relatively easy (show by sign-reversing involution that non-injective embeddings do not contribute to the total sum).

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Corollary

$(-1)^k \, \mathrm{Ch}_\mu$ *is a polynomial with nonnegative coefficients in* $\boldsymbol{p}$ *and* $-\boldsymbol{q}$ *(and thus in* $\boldsymbol{p}$ *and* $-\boldsymbol{r}$*).*

The formula is also suited for finding upper bounds for characters (F., Śniady, 2007).

A formula for normalized characters - sketch of proof

Use vanishing characterization: we define

$$F = \sum_{\substack{\sigma,\tau \\ \sigma\ \tau=\pi}} (-1)^{\tau} N_{\sigma,\tau} = \sum_{\substack{\sigma,\tau \\ \sigma\ \tau=\pi}} (-1)^{\tau} \widetilde{N_{\sigma,\tau}}$$

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2. That $F(\lambda) = 0$ if $|\lambda| < |\mu|$ is obvious from the second expression: indeed all $\widetilde{N_{\sigma,\tau}}(\lambda)$ are zero since $G(\sigma,\tau)$ has more edges than $G(\lambda)$.
3. The component of F of degree $|\mu|$ comes from the term $\sigma = \pi$ and $\tau = \mathrm{id}$ (in the first expression). But in this term is

$$N_{\pi,\mathrm{id}}(\lambda) = p_\mu(\lambda),$$

so that the top-component of F is p_μ as wanted.

Connection with maps combinatorics

- Reminder: if π has type μ, then $\mathrm{Ch}_\mu = \sum_{\substack{\sigma,\tau \\ \sigma\,\tau=\pi}} (-1)^\tau N_{\sigma,\tau}$.

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- But the set $\{\sigma, \tau;\ \sigma\,\tau = \pi\}$ is in bijection with maps (=graphs embedded in oriented surfaces) with prescribed face-degree. In other terms, Ch_μ is a signed weighted enumeration of maps.
- Maps with prescribed face-degree are better understood in the unicellular case, corresponding to $\mu = (k)$. For example, using a bijection of Chapuy ('11), we get this suprising relation:

$$\left(\sum_{\substack{i \geq 1 \\ h \geq 0}} \frac{p_i}{(2h+1)!} \frac{\partial^{2h+1}}{\partial p_i^{2h+1}} + \frac{r_i}{(2h+1)!} \frac{\partial^{2h+1}}{\partial r_i^{2h+1}} \right) \mathrm{Ch}_{(k)} = (k+1)\, \mathrm{Ch}_{(k)}\,.$$

Representation-theoretical consequence/interpretation? similar formulas for several parts?

Transition

Multirectangular coordinates –
expansion of shifted Schur functions

The falling factorial basis

Reminder: $s^\star_\mu(\lambda) = 0$ if $\lambda \not\supset \mu$.

$\rightarrow$ we cannot expect $s^\star_\mu$ to expand positively on multirectangular coordinates since it vanishes on a lot on Young diagrams (i.e. for a lot of positive specializations of multirectangular coordinates).

The falling factorial basis

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We introduce the falling factorial basis of multirectangular coordinates

$$(p_1 \downharpoonleft a_1) \dots (p_m \downharpoonleft a_m)(r_1 \downharpoonleft b_1) \dots (r_m \downharpoonleft b_m).$$

(Basis of polynomial ring in $\boldsymbol{p}$ and $\boldsymbol{r}$ when $a_1, \dots, a_m, b_1, \dots, b_m$ runs over lists of nonnegative integers.)

A nonnegative expansion for shifted Schur functions

Theorem (Alexandersson, F., 2017)

$s^\star_\mu$ expands positively on the falling factorial basis of multirectangular coordinates.

Corollary

$s^\star_\mu(x_1, \dots, x_n)$ expands positively on the basis

$$\Big((x_1 - x_2)_{b_1} \cdots (x_{\ell-1} - x_\ell)_{b_\ell - 1} (x_\ell)_{b_\ell} \Big)_{b_1,\dots,b_\ell \geq 0}.$$

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🙂 lifts to the polynomial $s^\star_\mu$ the nonnegativity of $s^\star_\mu(\lambda)$, for all partitions λ.

🙁 No combinatorial interpretation (although the coefficients of $|\mu|!\, s^\star_\mu$ are integers).

Main step of proofs

Using the previous formula for Ch_μ and the relation $s^\star_\mu = \sum_\nu \chi^\mu_\nu \, \mathrm{Ch}_\nu$, we get

$$s^\star_\mu = \frac{1}{k!} \sum_{\sigma,\tau \in \mathfrak{S}_k} \chi^\mu(\sigma\tau)\,(-1)^\tau\, N_{\sigma,\tau}.$$

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From there, we can extract the coefficient in FF-basis and reduce their nonnegativity to the one of

$$B^\mu_{S,T} = \sum_{\substack{\sigma \in \mathfrak{S}_S \\ \tau \in \mathfrak{S}_T}} \chi^\mu(\sigma\tau)\,(-1)^\tau,$$

for set-partitions S and T.

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We conclude with representation-theoretic arguments.

Transition

Quasi-symmetric functions
on Young diagrams

In which algebra lives N_G?

Expressing the shifted symmetric functions Ch_μ and $s^\star_\mu$ in terms of N_G gives nice expression.

But N_G is not shifted symmetric!

In the following slides, we study $Q\Lambda^\star := \mathrm{Span}(N_G)$ and connect it with quasi-symmetric functions.

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Definition

A polynomial F is symmetric if
for any exponents $a_1, \dots, a_r$, the coefficients of $x_{i_1}^{a_1} \dots x_{i_r}^{a_r}$ is the same for all repetition-free sequences $i_1, \dots, i_r$.

Example: $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_2^2 x_1 + x_3^2 x_1 + x_3^2 x_2 + \dots$

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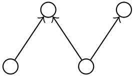
Definition

A polynomial F is quasi-symmetric if
for any exponents $a_1, \dots, a_r$, the coefficients of $x_{i_1}^{a_1} \dots x_{i_r}^{a_r}$ is the same for all increasing sequences $i_1, \dots, i_r$.

Example: $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + 2x_2^2 x_1 + 2x_3^2 x_1 + 2x_3^2 x_2 + \dots$

Detour: Quasi-symmetric function of an acyclic graph

Take an unlabelled acyclic directed graph $G_{\text{ex}} =$  .

Definition

A function $f : V_G \to \mathbb{N}$ is order-preserving if

$$(i,j) \in E_G \;\Rightarrow\; f(i) \leq f(j).$$

We consider the multivariate generating function in $x_1, x_2, \dots$

$$\Gamma(G) = \sum_{\substack{f:V\to\mathbb{N} \\ f \text{ order-preserving}}} \prod_{v\in V} x_{f(v)} \quad \in \mathrm{QSym}\,.$$

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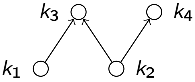
Take an unlabelled acyclic directed graph $G_{\text{ex}} =$ 

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$$(i,j) \in E_G \ \Rightarrow\ f(i) \leq f(j).$$

We consider the multivariate generating function in $x_1, x_2, \dots$

$$\Gamma(G) = \sum_{\substack{f: V \to \mathbb{N} \\ f \text{ order-preserving}}} \prod_{v \in V} x_{f(v)} \quad \in \mathrm{QSym}\,.$$

On the example above:

$$\Gamma(G_{\text{ex}}) = \sum_{\substack{k_1,k_2,k_3,k_4 \\ k_1 \leq k_3,\ k_2 \leq k_3,\ k_2 \leq k_4}} x_{k_1} x_{k_2} x_{k_3} x_{k_4}.$$

It is a quasisymmetric function (studied by Stanley, Gessel, ...).

Quasisymmetric functions as functions on Young diagrams

Theorem (Aval, F., Novelli, Thibon, 2015)

There is an isomorphism $\mathrm{QSym} \simeq Q\Lambda^\star = \mathrm{Span}(N_G)$ *such that*

- $\mathrm{Sym} \subset \mathrm{QSym}$ *is mapped to* $\Lambda^\star \subset Q\Lambda^\star$*;*
- *for bipartite graphs* G*, the function* $\Gamma(G)$ *is mapped to* N_G*.*

(Bipartite graphs are seen as acyclic graphs, by orienting edges from white to black.)

We have an "explicit" construction of the isomorphism using the virtual alphabet framework.

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As a consequence,

$$\underset{\text{is shifted symmetric}}{\sum c_G N_G} \quad \Leftrightarrow \quad \underset{\text{is symmetric}}{\sum c_G \Gamma(G)}$$

$\rightarrow$ more standard problem in symmetric function literature.

Conclusion

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- Seeing shifted symmetric functions as functions on Young diagrams leads to considering them in terms of various sets of coordinates;
- Nice combinatorics when using multirectangular coordinates;
- Tomorrow we will discuss Jack (and Macdonald) analogues of Ch_μ and $s^\star_\mu$; the extension of the positivity results shown today are only conjectured!