Patterns in meandric systems and tree-indexed sums of Catalan numbers

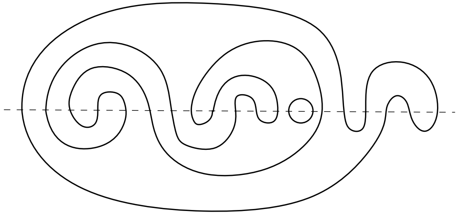
Valentin Féray
joint work with Paul Thévenin and Alin Bostan

CNRS, Université de Lorraine

Permutation Patterns
St Andrews, July 7th, 2025

Objects of interest: meandric systems...

A meandric system is a pair of two non-crossing pair-partitions.



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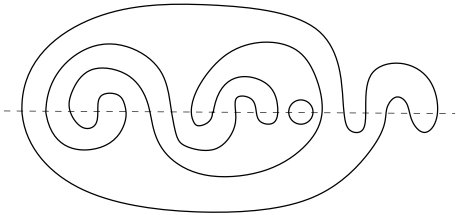

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But many questions involving connected components are hard (and interesting! links with percolation theory, quantum field theory, ...):

Conjecture (Di Francesco–Golinelli–Guitter, '00)

The number of connected meandric systems (a.k.a. **meanders**) of size n behaves asymptotically as $C\,A^n n^{-\alpha}$, with $\alpha = (29+\sqrt{145})/12 \approx 3.42$.

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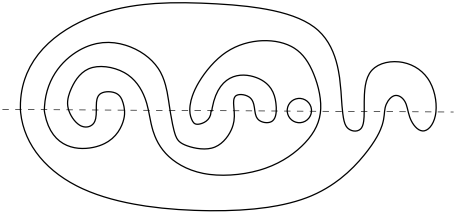

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But many questions involving connected components are hard (and interesting! links with percolation theory, quantum field theory, ...):

Conjecture (Borga–Gwynne–Park, '23)

The largest component of a uniform random meandric system has size $n^{\beta+o_P(1)}$, where $\beta = \frac{1}{2}(3-\sqrt{2}) \approx 0.79$.

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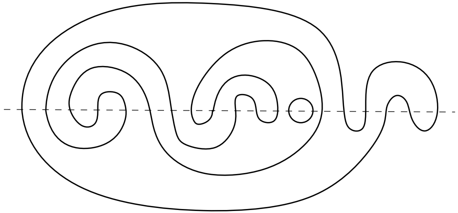

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But many questions involving connected components are hard (and interesting! links with percolation theory, quantum field theory, ...):

Theorem (F.–Thévenin '23, conjectured by Goulden–Nica–Puder and Kargin '20)

The number of connected components of a uniform random meandric system is $(\kappa + o_P(1))n$, for some constant $\kappa \approx 0.23$.

Objects of interest: meandric systems... **and their patterns**

All these questions can be formulated in terms of the random variable $\left|\boldsymbol{C_i(M_n)}\right|$, i.e. the size of the component of a uniform random element i in a uniform random meandric system M_n of size n.

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All these questions can be formulated in terms of the random variable $\boldsymbol{|C_i(M_n)|}$, i.e. the size of the component of a uniform random element i in a uniform random meandric system M_n of size n.

Our contribution: define a notion of shape/pattern of the component if 0, and compute, for a given S,

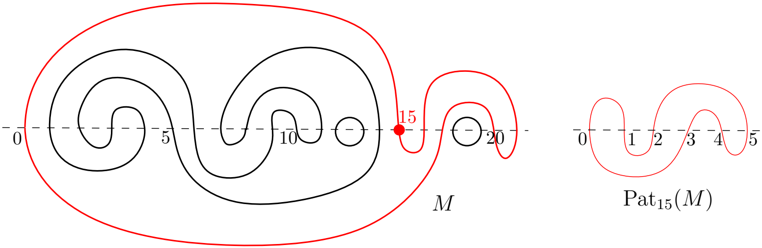
$$\lim_{n\to+\infty} \mathbb{P}(C_i(M_n) \simeq S).$$

Note: The probability $\mathbb{P}(|C_i(M_n)|)$ is then a finite sum of "shape probabilities".

Definition: patterns in meandric systems

Let M be a meandric system, i an element of M, and set $k = |C_i(M)|$.

The pattern $\mathrm{Pat}_i(M)$ of i in M is obtained by relabelling the vertices of $C_i(M)$ with the unique increasing bijection $V(C_i(M)) \to \{0, \ldots, 2k-1\}$.

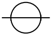


Note: $\mathrm{Pat}_i(M)$ is a meander.

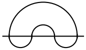
Result 1 (F.–Thévenin '23): an expression of $\lim_{n\to+\infty}\mathbb{P}(C_i(M_n)\simeq S)$ as a mutli-indexed sum of "normalized" Catalan numbers $\widetilde{\mathrm{Cat}}_k = 4^{-k}\,\mathrm{Cat}_k$.

Examples

$$\lim_{n\to+\infty}\mathbb{P}\Big(C_i(M_n)\simeq \quad\Big) = \frac{1}{8}\sum_{\ell=0}^{\infty}\widetilde{\mathrm{Cat}}_\ell^2 \tag{1}$$



$$\lim_{n\to+\infty}\mathbb{P}\Big(C_i(M_n)\simeq \quad\Big) = \frac{1}{64}\cdot\sum_{\ell_1,\ell_2,\ell_3\geq 0}\widetilde{\mathrm{Cat}}_{\ell_1}\,\widetilde{\mathrm{Cat}}_{\ell_2}\,\widetilde{\mathrm{Cat}}_{\ell_3}\,\widetilde{\mathrm{Cat}}_{\ell_1+\ell_3} \tag{2}$$



Informal statement of the results

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Result 2 (Bostan–F.–Thévenin '25): an algorithm computing these sums (in particular, they are always polynomials in $1/\pi$).

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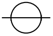
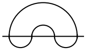
$\lim_{n\to+\infty}\mathbb{P}\Big(C_i(M_n)\simeq$  $\Big) = \frac{1}{8}\sum_{\ell=0}^{\infty}\widetilde{\mathrm{Cat}}_\ell^{\,2} = \frac{2}{\pi}-\frac{1}{2}\approx 0.137$ (1)

$\lim_{n\to+\infty}\mathbb{P}\Big(C_i(M_n)\simeq$  $\Big) = \frac{1}{64}\cdot\sum_{\ell_1,\ell_2,\ell_3\ge 0}\widetilde{\mathrm{Cat}}_{\ell_1}\widetilde{\mathrm{Cat}}_{\ell_2}\widetilde{\mathrm{Cat}}_{\ell_3}\widetilde{\mathrm{Cat}}_{\ell_1+\ell_3}$

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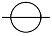
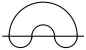
$$\lim_{n\to+\infty}\mathbb{P}\Big(C_i(M_n)\simeq$$  $$\Big) = \frac{1}{8}\sum_{\ell=0}^{\infty}\widetilde{\mathrm{Cat}}_\ell^{\,2} = \frac{2}{\pi}-\frac{1}{2}\approx 0.137 \qquad (1)$$

$$\lim_{n\to+\infty}\mathbb{P}\Big(C_i(M_n)\simeq$$  $$\Big) = \frac{1}{64}\cdot\sum_{\ell_1,\ell_2,\ell_3\geq 0}\widetilde{\mathrm{Cat}}_{\ell_1}\,\widetilde{\mathrm{Cat}}_{\ell_2}\,\widetilde{\mathrm{Cat}}_{\ell_3}\,\widetilde{\mathrm{Cat}}_{\ell_1+\ell_3}$$
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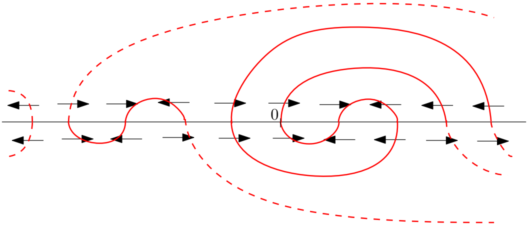
Next few slides: I'll explain Result 1.

The Uniform Infinite Meandric System, or Infinite Noodle (Soup)

Definition (UIMS)

Draw two bi-infinite sequences of i.i.d. left/right arrows and connect them in the unique non-crossing way. The resulting configuration is called Infinite Noodle, and denoted M_∞.



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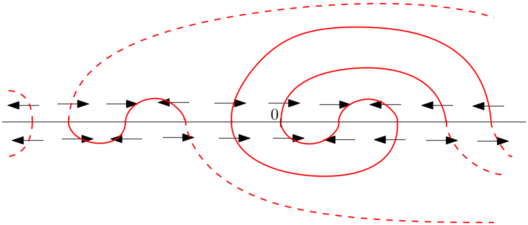


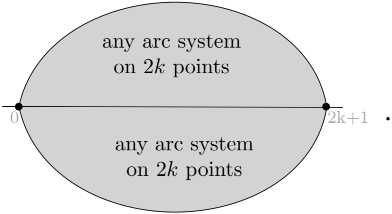
Proposition (F.–Thévenin, '23)

$$\lim_{n\to+\infty} \mathbb{P}(C_i(M_n) \simeq S) = \mathbb{P}(C_0(M_\infty) \simeq S).$$

Note: whether $C_0(M_\infty)$ is a.s. finite or not is an open question.

Warmup: $S =$ 

Up to changing the place of 0, a realization of M_∞ with $C_0(M_\infty) \simeq S$ looks like this:



Hence

$$\mathbb{P}[C_0(M_\infty) \simeq S] = 2 \sum_{k \geq 1} \mathrm{Cat}_k^2 \, 2^{-4k-4} = \frac{1}{8} \sum_{k \geq 1} \widetilde{\mathrm{Cat}}_k^2 .$$

More interesting: $S =$ 

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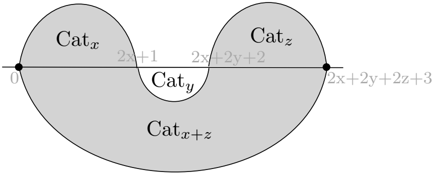

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Hence

$$\mathbb{P}[C_0(M_\infty) \simeq S] = 4 \sum_{x,y,z \geq 0} \mathrm{Cat}_x \,\mathrm{Cat}_y \,\mathrm{Cat}_z \,\mathrm{Cat}_{x+z} \, 2^{-4x-2y-4z-8}$$

$$= \frac{1}{64} \sum_{x,y,z \geq 0} \widetilde{\mathrm{Cat}}_x \,\widetilde{\mathrm{Cat}}_y \,\widetilde{\mathrm{Cat}}_z \,\widetilde{\mathrm{Cat}}_{x+z} \,.$$

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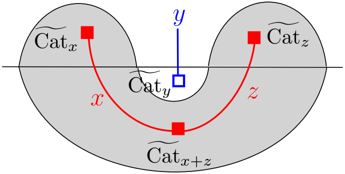

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Let us draw the "dual forest" of S. We observe that there is one summation index for each edge of the forest, and one Catalan factor for each vertex.

General S: tree indexed sums of Catalan numbers

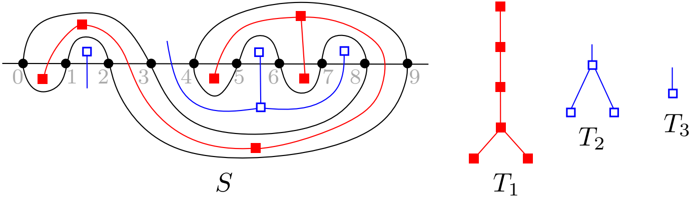
For a tree T, we set $\Sigma(T) := \sum_{(x_e) \in \mathbb{Z}_+^{E(T)}} \left(\prod_{v \in V(T)} \widetilde{\mathrm{Cat}}_{\sum_{e \ni v} x_e} \right).$

Proposition (F.–Thévenin '23)

For any meander S of size k, we have

$$\mathbb{P}(C_0(M_\infty) \simeq S) = 2^{-4k+1} k \prod_{i=1}^{d} \Sigma(T_i),$$

where the T_i's are the "dual trees" of the meander.



Computing $\Sigma(T)$ - main result

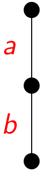
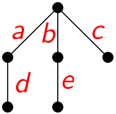
For a tree T, we set $\Sigma(T) := \sum_{(x_e)\in\mathbb{Z}_+^{E(T)}} \left(\prod_{v\in V(T)} \widetilde{\mathrm{Cat}}_{\sum_{e\ni v} x_e} \right).$

Theorem (Bostan–F.–Thévenin '25)

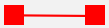
For any tree T, the sum $\Sigma(T)$ is a polynomial in $1/\pi$ of degree at most $|V_T|/2$.

Moreover, we provide an algorithm to compute these sums.

Computing $\Sigma(T)$ - examples

	$\Sigma(T) = \sum_{a,b \geq 0} \widetilde{\mathrm{Cat}}_a \widetilde{\mathrm{Cat}}_{a+b} \widetilde{\mathrm{Cat}}_b$ $= \boxed{8 - \frac{64}{3\pi}}$
	$\Sigma(T) = \sum_{\mathbb{Z}_+^5} \widetilde{\mathrm{Cat}}_{a+b+c} \widetilde{\mathrm{Cat}}_{a+d} \widetilde{\mathrm{Cat}}_{b+e} \widetilde{\mathrm{Cat}}_c \widetilde{\mathrm{Cat}}_d \widetilde{\mathrm{Cat}}_e$ $= \boxed{-128 - \frac{512}{9\pi} + \frac{1024}{\pi^2} + \frac{4096}{3\pi^3}}$

Mathematica (or Maple) can deal with the first example, but not with the second one!

$\Sigma($  $)$ and hypergeometric functions

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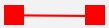
$\Sigma($  $)$ and hypergeometric functions

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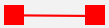
$$ {}_2F_1(a,b;c;z) := \sum_{n \geq 0} \frac{a^{\uparrow n} b^{\uparrow n}}{c^{\uparrow n}} \frac{z^n}{n!}, $$

where $u^{\uparrow n} := u(u+1)\cdots(u+n-1)$.

$\Sigma($  $)$ and hypergeometric functions

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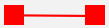
In fact, we have $\Sigma($  $) = 4 \cdot {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1\right) - 4$.

$\Sigma($  $)$ and hypergeometric functions

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Lemma (Gauss identity)

If $c - a - b > 0$, we have

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

Thus ${}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1\right) = \frac{4}{\pi}$ and $\Sigma($  $) = \frac{16}{\pi} - 4$.

$\Sigma($  $)$ and the quadratic recurrence

We want to compute $\Sigma($  $) = \sum_{x,y \in \mathbb{Z}_+} \mathrm{Cat}_x \, \mathrm{Cat}_y \, \mathrm{Cat}_{x+y} \, 16^{-x-y}$.

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Rewrite the sum using $Z = x + y$.

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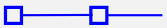
Again, this can be related to hypergeometric functions, namely

$\Sigma($  $)$ $= 8 - 8 \cdot {}_2F_1\left(-\tfrac{1}{2}, \tfrac{1}{2}; 2; 1\right)$

and Gauss identity allows to compute

$\Sigma($  $)$ $= 8 - \dfrac{64}{3\pi}.$

$\Sigma($  $)$: changing variables and manipulating inequalities

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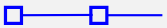
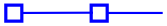
$\Sigma($  $)$: changing variables and manipulating inequalities

We want to compute $\Sigma($  $) = \sum_{x,y \in \mathbb{Z}_+} \mathrm{Cat}_x \, \mathrm{Cat}_{x+y} \, 4^{-2x-y}$.

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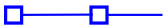
$\Sigma($  $) = \sum_{z \geq x \geq 0} \mathrm{Cat}_x \, \mathrm{Cat}_z \, 4^{-x-z}.$

$\Sigma(\square\text{—}\square\text{—})$: changing variables and manipulating inequalities

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$$\Sigma(\square\text{—}\square\text{—}) = \sum_{z \geq x \geq 0} \mathrm{Cat}_x \, \mathrm{Cat}_z \, 4^{-x-z}.$$

By symmetry, we also have

$$\Sigma(\square\text{—}\square\text{—}) = \sum_{x \geq z \geq 0} \mathrm{Cat}_x \, \mathrm{Cat}_z \, 4^{-x-z}.$$

and thus

$$2\Sigma(\square\text{—}\square\text{—}) = \sum_{x,z \geq 0} \mathrm{Cat}_x \, \mathrm{Cat}_z \, 4^{-x-z} + \sum_{\substack{x,z \geq 0 \\ x=z}} \mathrm{Cat}_x \, \mathrm{Cat}_z \, 4^{-x-z}$$

$$= \left(\sum_{x \geq 0} \mathrm{Cat}_x \, 4^{-x} \right)^2 + \Sigma(\blacksquare\text{—}\blacksquare) = 4 + \left(\frac{16}{\pi} - 4 \right) = \frac{16}{\pi}.$$

Overview of the algorithm computing $\Sigma(T)$

- We forget about edge variables, and use vertex-indexed variables constrained by one equality and inequalities.

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- The induction step uses inequality manipulations and the Catalan quadratic recurrence.

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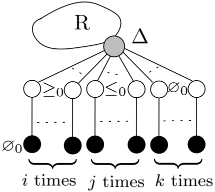
- We need to generalize the problem with shifts, more general inequalities, ...
- The base case uses the linear recurrence of Catalan numbers and hypergeometric function identity.
- The induction step uses inequality manipulations and the Catalan quadratic recurrence.
- The induction is intricate (see next slide)....

A linear system for long stars

$R \mid V^{\Delta}_{i,j,k} :=$ 

Lemma

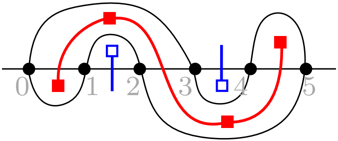
For any "rootstock" R, any decoration Δ and any $d \geq 2$, we have

$$\begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 2 & 1 \\ 0 & \dots & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} S(R \mid V^{\Delta}_{1,d-1,0}) \\ \vdots \\ S(R \mid V^{\Delta}_{d-1,1,0}) \end{pmatrix} = \begin{pmatrix} X_1 \\ \vdots \\ X_{d-1} \end{pmatrix} - \begin{pmatrix} S(R|V^{\Delta}_{0,d,0}) \\ 0 \\ \vdots \\ 0 \\ S(R|V^{\Delta}_{d,0,0}) \end{pmatrix},$$

where, for $1 \leq i \leq d-1$,

$X_i = S(R \mid V^{\Delta}_{i-1,d-1-i,2}) + 2S(R \mid V^{\Delta}_{i-1,d-1-i,1}) \cdot S($  $) + S(R \mid V^{\Delta}_{i-1,d-1-i,0}) \cdot S($  $)^2$

Thanks for your attention!

$S =$ 

$\Sigma($  $)$

$$= \sum_{x,y,z} \widetilde{\mathrm{Cat}}_x \, \widetilde{\mathrm{Cat}}_{x+y} \, \widetilde{\mathrm{Cat}}_{y+z} \, \widetilde{\mathrm{Cat}}_z$$

$$= -32 + \frac{64}{\pi} + \frac{128}{\pi^2}.$$

$\lim_{n\to+\infty} \mathbb{P}(C_0(M_\infty) \simeq S) = \frac{3}{2^{11}} \Sigma($  $) \Sigma($  $)^2$

$$= -\frac{3}{4} + \frac{3}{2\pi} + \frac{3}{\pi^2} \approx 0.031428$$