Patterns in meandric systems and tree-indexed sums of Catalan numbers

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This talk is based on joint work with Alin Bostan and Paul Thévenin

We study *meandric systems* of size n, which are collections of non-crossing loops intersecting the horizontal axis exactly at the points $0, \dots, 2n-1$ (up to continuous deformation fixing the horizontal axis). Combinatorially, meandric systems can be uniquely represented as a pair (P_1, P_2) of non-crossing pair-partitions; in particular there are Cat_n^2 meandric systems of size n, where Cat_n is the n-th Catalan number.

Given a number i in $\{1, \dots, 2n\}$ and a meander system $M = (P_1, P_2)$, we are interested in the "pattern" or "shape" (of the component) of i, defined as follows. We see M as mutligraph, where vertices are the integers from 0 to $2n-1$ and the edges are given by the pairs in P_1 and P_2. In this construction, each vertex has degree 2 and the connected components are closed loops. Let us write $C_i(M)$ for the component of i, and $2k$ for its number of vertices. Then the pattern $\mathrm{Pat}_i(M)$ of i in M is obtained by relabelling the vertices of $C_i(M)$ with the unique increasing bijection $V(C_i(M)) \to \{0, \dots, 2k-1\}$. An example is given on Fig. 1. By construction, $\mathrm{Pat}_i(M)$ is always a meandric system with a single connected component, aka a meander.

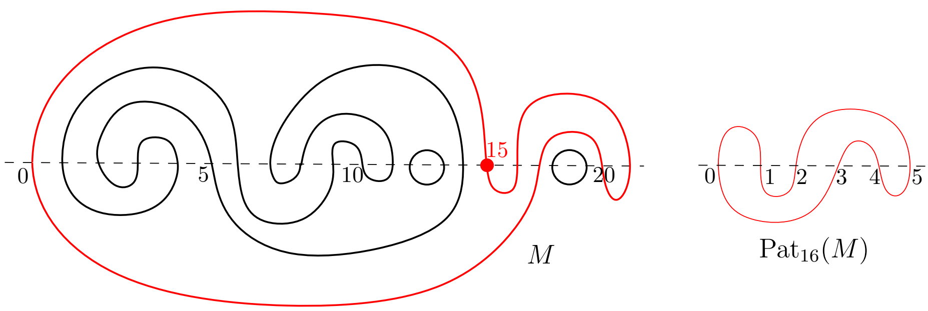

Figure 1: Left: a meandric system M. Its vertices of the meandric system are labelled from 0 to 21, but to make the picture more readable, we only typed every fifth label. We look at the pattern of the red vertex $i = 15$ in M. The component of i is drawn in red; it contains the vertices 0, 16, 17, 18, 20 and 21. To get the pattern $\mathrm{Pat}_i(M)$ (shown on the right part of the picture), we relabel them as 0, 1, 2, 3, 4 and 5 respectively.

Given a fixed meander M_0, we are interested in the large n limit of the probability $\mathbb{P}(C_i(M_n) = M_0)$, where M_n is a uniform random meandric system of size n, and i is a uniform random integer in $\{0, \dots, 2n-1\}$, independent from M_n. We call this quantity the *asymptotic frequency* of M_0. For example, we can prove that

$$\lim_{n\to+\infty} \mathbb{P}\left(C_i(M_n) = \quad\right) = \frac{1}{8}\sum_{\ell=0}^{\infty} \mathrm{Cat}_\ell^2\, 2^{-4\ell} = \frac{2}{\pi} - \frac{1}{2} \approx 0.137 \tag{1}$$

$$\begin{aligned}\lim_{n\to+\infty} \mathbb{P}\left(C_i(M_n) = \quad\right) &= \frac{1}{64}\cdot \sum_{\ell_1,\ell_2,\ell_3\geq 0} \mathrm{Cat}_{\ell_1}\,\mathrm{Cat}_{\ell_2}\,\mathrm{Cat}_{\ell_3}\,\mathrm{Cat}_{\ell_1+\ell_3}\, 2^{-4\ell_1-2\ell_2-4\ell_3} \\ &= \frac{1}{4} - \frac{2}{3\pi} \approx 0.038. \end{aligned}\tag{2}$$

Our first result, proved in [1] and generalizing the first equality in (1), (2), is a combinatorial summatory expression for the asymptotic frequency of M_0. To state it, we need to introduce some notation. First, for a meander M_0, we consider its dual trees as shown on Fig. 2: their vertices are the bounded regions delimited by the meander and the horizontal axis, and two vertices are connected, if the corresponding regions share an interval of the horizontal axis as boundary. We also add an half-edge to a vertex whenever the corresponding region is connected to an unbounded region.

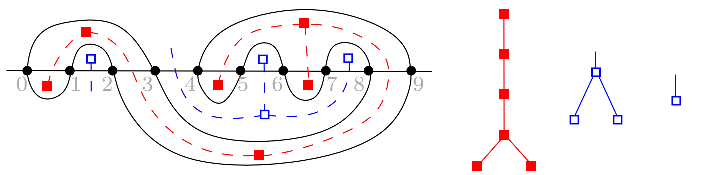

Figure 2: A meander and its dual trees.

Second, given a tree T, possibly with an half-edge, we consider the sum

$$S(T) := \sum_{(x_e)\in\mathbb{Z}_+^{E(T)}} \left(\prod_{v\in V(T)} \mathrm{Cat}_{X_v}\, 4^{-X_v}\right), \tag{3}$$

where for a vertex v, we set $X_v = \sum_{e\ni v} x_e$ (i.e. we sum x_e over all edges incident to v). For example, if P_2 (resp. P_2^h) is the path with two vertices linked by an edge, (resp. with an additional half-edge), we have by definition:

$$\begin{aligned} S(P_2) &= \sum_{x\in\mathbb{Z}_+} \left(\mathrm{Cat}_x\, 4^{-x}\right)^2 \\ S(P_2^h) &= \sum_{x_1,x_2\in\mathbb{Z}_+} \mathrm{Cat}_{x_1}\, 4^{-x_1}\,\mathrm{Cat}_{x_1+x_2}\, 4^{-x_1-x_2}. \end{aligned}$$

Proposition 1. *For any meander M_0 of size k, we have*

$$\lim_{n\to+\infty} \mathbb{P}\big(C_i(M_n) = M_0\big) = 2^{-4k+1} k\, S(T_1)\cdots S(T_m) \tag{4}$$

where $T_1, \ldots, T_m$ are the dual trees of M_0.

Our second result, proved in [2] and generalizing the second equality in (1), (2), is a structural result on these numbers.

Theorem 2. *For any tree T with zero or one half-edge, the multiple sum $S(T)$ is a polynomial in $1/\pi$. Consequently, the asymptotic frequency $\lim_{n\to+\infty} \mathbb{P}\big(C_i(M_n) = M_0\big)$ of any meander M_0 is a polynomial in $1/\pi$.*

The proof is constructive and uses both the linear and the quadratic recurrences of Catalan numbers, together with properties of the hypergeometric functions ${}_2F_1$.

We also get a compact expression for stars. Namely, if T_s is the tree with a central vertex connected with s leaves, then

$$S(T_{s+3}) = \frac{64}{\pi} \cdot \left(\sum_{k=0}^{s} \binom{s}{k} \cdot \frac{1}{(2k+1)\,(2k+3)\,(2k+5)} \right).$$

References

[1] V. Féray and P. Thévenin. Components in meandric systems and the infinite noodle. *Int. Math. Res. Not.*, 2023(14):12538–12560, 2023.

[2] A. Bostan, V. Féray and P. Thévenin. Tree-indexed sums of Catalan numbers. Soon on arXiv, 2025.