Patterns and random permutations II

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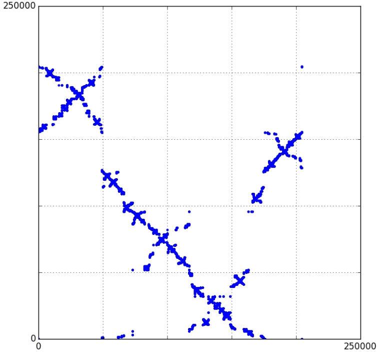
Universität Zürich[UZH]

Content of this lecture

Study the limit of separable permutations.



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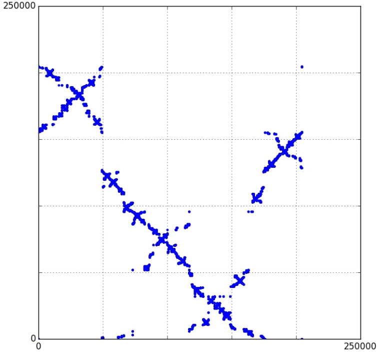
Extend the approach to substitution-closed classes
→ the limit of separable permutations is in some sense universal.



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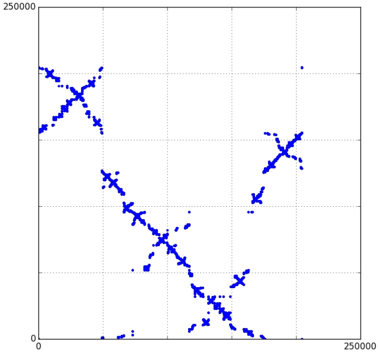

Tools: yesterday's convergence criterion + analytic combinatorics.

Separable permutations

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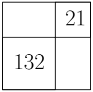
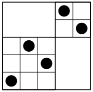
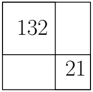
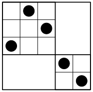
$\oplus[132, 21] =$ 

$=$  $= 13254$

$\ominus[132, 21] =$ 

$=$  $= 35421$

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The class of separable permutations is the smallest sets of permutations containing 1 and stable by $\oplus$ and $\ominus$.

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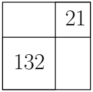
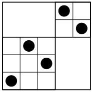
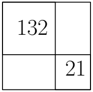
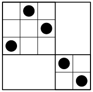
$\oplus[132, 21] =$ 

$=$  $= 13254$

$\ominus[132, 21] =$ 

$=$  $= 35421$

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Separable permutations pop up in connection with: sorting algorithms, bootstrap percolation, polynomial interchanges, ...

Tree description

Example of a separable permutation (obtained by iterating $\oplus$ and $\ominus$ operations:

$$\mathrm{perm}(t,\varepsilon) = \oplus\Big[\oplus\big[\ominus[1,1,1],1\big],1,\ominus[1,1]\Big] = \oplus\big[\oplus[321,1],1,21\big]$$
$$= \oplus[3214,1,21] = 3214576.$$

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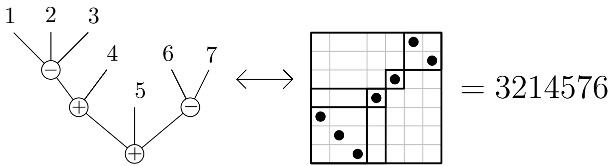

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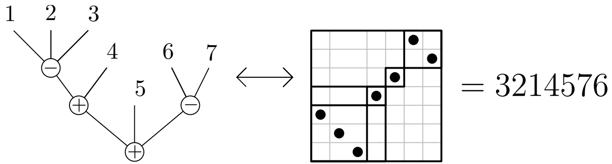
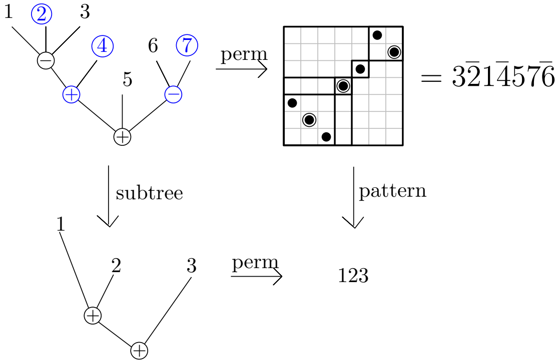


⚠ This tree is not unique! To get uniqueness, impose alternating signs.

Patterns and trees

The following diagram commutes:



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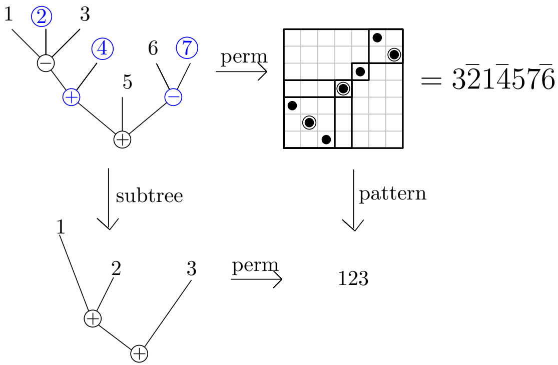

Consequence: let $\sigma = \mathrm{perm}(T, S)$ a permutation. Take k distinct leaves uniformly at random in T and call $(\boldsymbol{t}, \varepsilon)$ the corresponding random signed subtree. Then

$$\widetilde{\mathrm{occ}}(\pi, \sigma) = \mathbb{P}(\mathrm{perm}(\boldsymbol{t}, \varepsilon) = \pi).$$

Convergence of separable permutations

Let $\boldsymbol{\sigma}^{(n)}$ be a uniform random separable permutation of size n.

Theorem (BBFGP,'16)

$\mu_{\boldsymbol{\sigma}^{(n)}}$ tends towards a non-deterministic permuton $\boldsymbol{\mu}$, whose pattern densities $\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\mu})$ are constructed below.

Fix a pattern π of size k. Let $(\boldsymbol{\mathcal{T}}, \boldsymbol{S})$ be the continuous Brownian tree with i.i.d. balanced signs on its branching points. Take k points uniformly at random in $\boldsymbol{\mathcal{T}}$ and extract the corresponding signed subtree $(\boldsymbol{t}, \boldsymbol{\varepsilon})$. Then

$$\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\mu}) = \mathbb{P}\Big[\,\mathrm{perm}(\boldsymbol{t}, \boldsymbol{\varepsilon}) = \pi \Big| (\boldsymbol{\mathcal{T}}, \boldsymbol{S})\Big].$$

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Intuition: the tree encoding $\boldsymbol{\sigma}^{(n)}$ converges to $\boldsymbol{\mathcal{T}}$ (as many families of random trees) and the signs of the extracted subtrees are asymptotically independent.

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- $\boldsymbol{\mu}$ is called the Brownian separable permuton.
- One can also construct directly $\boldsymbol{\mu}$ from $(\boldsymbol{\mathcal{T}}, \boldsymbol{S})$ (Maazoun, '17+)

Transition

Substitution-closed classes
and universality of
the Brownian separable permuton

Substitution in permutations

Definition

Let θ be a permutation of size d and $\pi^{(1)}, \dots, \pi^{(d)}$ be permutations. The diagram of the permutation $\theta[\pi^{(1)}, \dots, \pi^{(d)}]$ is obtained by replacing the i-th dot in the diagram of θ with the diagram of $\pi^{(i)}$ (for each i).

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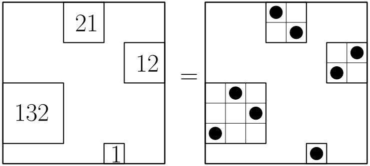

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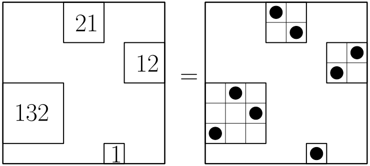

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Note: $\oplus$ (resp. $\ominus$) are substitution in increasing (resp. decreasing) permutations, so separable permutations form a (the simplest) substitution-closed class.

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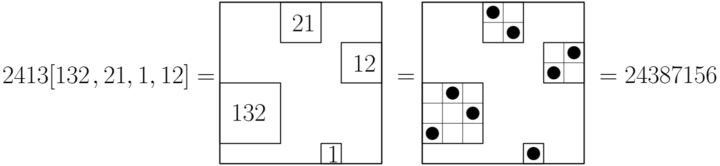

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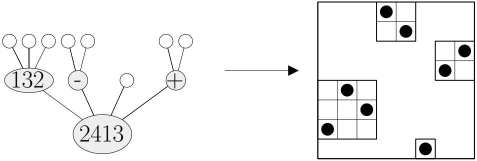
Def: a permutation is simple if it can not be written as substitution of smaller permutations ($\sim \frac{n!}{e^2}$ permutations of size n).
Then $\mathrm{Av}(\tau_1, \cdots, \tau_r) \Leftrightarrow \tau_1, \cdots, \tau_r$ are simple.

Tree representation in substitution closed classes (Albert, Atkinson, '05)

As separable permutations, permutations in a substitution closed class $\mathcal{C}$ can be represented by "substitution trees":



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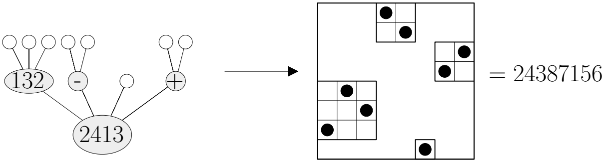


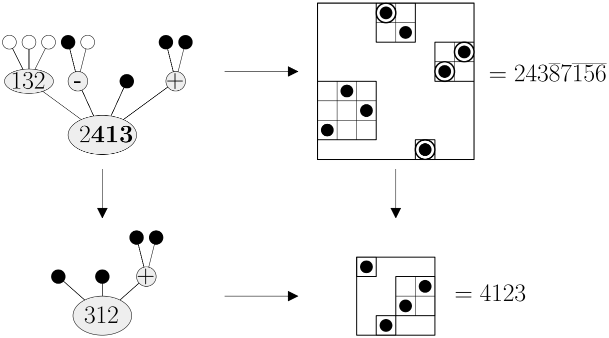
The tree is unique (and then called canonical tree) if we require:

- no adjacent $\oplus$ (resp. $\ominus$) nodes;
- permutations labeling the nodes are simple.

$\rightarrow$ the set $\mathcal{S}$ of simple permutations in $\mathcal{C}$ will play a crucial role.

Subtree-pattern equivalence in substitution-closed classes

Again, we have a commutative diagram



Universality of the (biased) Brownian separable permuton

Theorem (BBFGMP, '17)

Let $\mathcal{C}$ be a substitution-closed class whose set of simple permutations $\mathcal{S}$ has generating function $S(z) = \sum_{\alpha \in \mathcal{S}} z^{|\alpha|}$. Assume

$$R_S > 0 \quad \textit{and} \quad S'(R_S) > \frac{2}{(1+R_S)^2} - 1. \tag{H1}$$

R_S: radius of convergence of $S(z)$.

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For every $n \geq 1$, let $\boldsymbol{\sigma}_n$ be a uniform permutation in $\mathcal{C}$. The sequence $(\mu_{\boldsymbol{\sigma}_n})_n$ tends to the biased Brownian separable permuton $\boldsymbol{\mu}^{(p)}$ for some explicit parameter p in $[0,1]$.

Biased Brownian separable permuton $\boldsymbol{\mu}^{(p)}$: $\widetilde{\text{occ}}(\pi, \boldsymbol{\mu}^{(p)})$ is as $\widetilde{\text{occ}}(\pi, \boldsymbol{\mu})$, except that the signs on the branching points of $\boldsymbol{\mathcal{T}}$ are $+$ with proba p (independently from each other).

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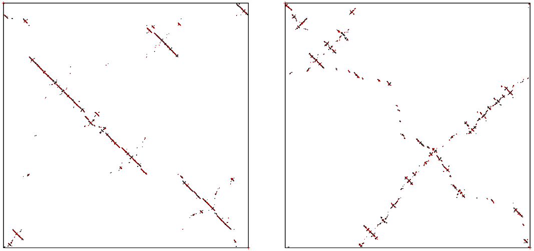
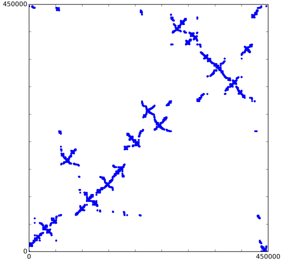
- universality phenomenon: the limit only depends on $\mathcal{S}$ through a single parameter p (in practice, always closed to $1/2$).
- intuition: tree encoding $\boldsymbol{\sigma}_n$ tends towards the continuous Brownian tree.

Pictures



Simulation of biased Brownian permutons for p=.2 and p=.45



Simulation of a permutation in the substitution-closed class with simples 2413, 3142 and 24153.

Other limiting behaviours

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Note: we always assume $R_S > 0$, which exclude only the class of all permutations.

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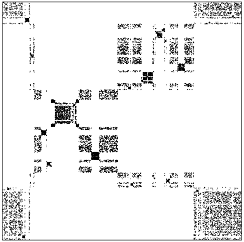
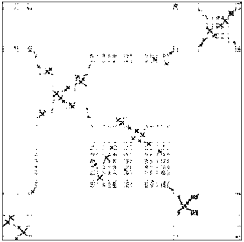
Intuition: in case 2, the tree encoding $\boldsymbol{\sigma}_n$ has one vertex of very large degree. In case 3b., it tends towards a stable tree.

(Cases 2, 3a and 3b require additional technical hypotheses.)

Pictures (stable permutons)





Simulation of stable permutons of parameter $\delta = 1.1$ and $\delta = 1.5$
(Stable permutons depend deeply on the set of simples; here we assume that a uniform large random simple permutation is close to a uniform random permutation)

$\emptyset$

We do not know substitution-closed classes which fits in case 3b.

Transition

Ideas of proofs

Reminder: expectations are enough

Enough to prove that, for any π,

$$\mathbb{E}\left[\,\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)\right] \to \mathbb{E}\left[\,\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\nu})\right],$$

where $\boldsymbol{\nu}$ is the targeted limit random permuton.

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On both side,

$$\widetilde{\mathrm{occ}}(\pi, \dots) = \sum_t \widetilde{\mathrm{occ}}(t, \dots),$$

where the sum runs over substitution trees of π.

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The right-hand side is explicit (from the theory of random trees):

Brownian case $\widetilde{\mathrm{occ}}(t, \boldsymbol{\mu}^{(p)}) = \frac{\mathbf{1}[t \text{ binary}]}{\mathrm{Cat}_{k-1}} p^{\#+(t)}(1-p)^{\#-(t)}$;

stable case more complicated formulas (not only binary trees appear).

How to evaluate $\mathbb{E}\big[\,\widetilde{\mathrm{occ}}(t, \boldsymbol{\sigma}_n)\big]$?

Do combinatorics! Recall that permutation in $\mathcal{C}$ are uniquely represented by canonical trees, then

$$\mathbb{E}\big[\,\widetilde{\mathrm{occ}}(t, \boldsymbol{\sigma}_n)\big] = \frac{\mathrm{Num}_n^{(t)}}{\mathrm{Den}_n},$$

with

$$\mathrm{Den}_n = \#\{\text{canonical trees}\}$$

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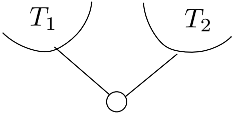
To get the asymptotics, we use analytic combinatorics.

Scratch course in analytic combinatorics

We want to evaluate asymptotically some sequence c_n of numbers of combinatorial objects. Consider the generating function $C(z) = \sum c_n z_n$. Two steps:

1. write equation for the generating series $C(z)$, based on decomposition of the objects. Example, for binary trees,



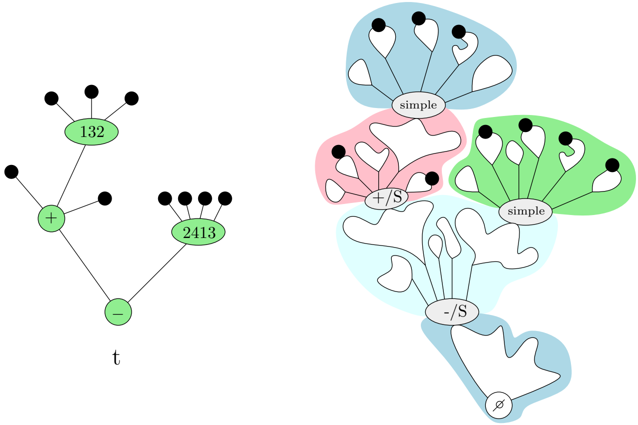
$$\Rightarrow \; C(z) = 1 + zC(z)^2.$$

2. Study the behaviour of $C(z)$ near the smallest singularity ρ. Then getting the asymptotic of c_n is automatic: e.g.,

$$C(z) = A\left(1 - \tfrac{z}{\rho}\right)^{\beta}(1 + o(1)) \;\Rightarrow\; c_n = \frac{A}{\rho^n}\,\frac{n^{-(\beta+1)}}{\Gamma(-\beta)}.$$

(Under technical additional assumptions.)

Combinatorial decomposition of canonical trees with marked leaves inducing a given t



The white pieces are trees with zero or one marked leaf and some conditions (to avoid creating adjacent $\oplus$ by gluing).

Translating that into equations (1/2)

Equations for the white pieces:

- One implicit equation

$$T_{\mathrm{not}\oplus} = z + \frac{T_{\mathrm{not}\oplus}^2}{1 - T_{\mathrm{not}\oplus}} + S\left(\frac{T_{\mathrm{not}\oplus}}{1 - T_{\mathrm{not}\oplus}}\right).$$

- Other series are expressed in terms of this one

$$T = \frac{T_{\mathrm{not}\oplus}}{1 - T_{\mathrm{not}\oplus}};$$

$$T^+ = \frac{1}{1 - WS'(T) - W - S'(T)};$$

$$T^+_{\mathrm{not}\ominus} = \frac{1}{1 + W}\, T^+;$$

$$T^+_{\mathrm{not}\oplus} = (WS'(T) + W + S'(T))\, T^+_{\mathrm{not}\ominus}$$

where $W = (\frac{1}{1 - T_{\mathrm{not}\oplus}})^2 - 1$.

Translating that into equations (2/2)

Equation for $\mathrm{Num}^{(t)}(z) = \sum \mathrm{Num}_n^{(t)} z^n$:

$$\mathrm{Num}^{(t)}(z) = z^k \sum_{V_s} T^{\text{type of root}} \prod_{v \in \mathrm{Int}(t)} A_v,$$

where

$$A_v = \begin{cases} \mathrm{Occ}_{\theta_v}(T)\,(T')^{d'_v}(T^+)^{d_v^+}(T^-)^{d_v^-} & \text{if } v \in V_s\,, \\ \left(\frac{1}{1-T_{\mathrm{not}\oplus}}\right)^{d_v+1}(T'_{\mathrm{not}\oplus})^{d'_v}(T^+_{\mathrm{not}\oplus})^{d_v^+}(T^-_{\mathrm{not}\oplus})^{d_v^-} & \text{if } v \notin V_s \text{ and } \theta_v = \oplus\,, \\ \left(\frac{1}{1-T_{\mathrm{not}\ominus}}\right)^{d_v+1}(T'_{\mathrm{not}\ominus})^{d'_v}(T^+_{\mathrm{not}\ominus})^{d_v^+}(T^-_{\mathrm{not}\ominus})^{d_v^-} & \text{if } v \notin V_s \text{ and } \theta_v = \ominus\,. \end{cases}$$

Second step: singularity analysis

1. Find singularity exponents the singular part of all these series is $\mathrm{cst}(1-\frac{z}{\rho})^{\beta}(1+o(1))$ where β is:

	Brownian case	stable case	degenerate case
simple permutations	analytic	$\delta \in (1,2)$	$\delta > 1$
canonical trees	$1/2^*$	$1/\delta$	δ
trees with one marked leaf	$-1/2$	$\frac{1}{\delta}-1$	$\delta-1$
$\mathrm{Num}^{(t)}(z)$	$-(e+1)/2$	0	$\sum_v (\delta-d_v)^-$

e: number of edges of t; d_v: number of children of v; $x = \min(x,0)$.

$*$: this $1/2$ exponent is classical for series defined through analytic implicit equations.

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3. Compute constants for such trees. . .

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4. Perspectives:
   - construction and properties (like the Hausdorff diemnsion) of the stable permuton or its pattern densities;
   - study local convergence of separable permutations/permutations in substitution-closed classes (what do we see around a random point?)
   - limits of uniform permutations in other classes/non-uniform model of random permutations;