Patterns and random permutations I

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Introduction

The last two lectures focus of random permutations:

- Classical questions: look at some statistics, like the number of cycles (of given length), longest increasing subsequences, ...
(usually for uniform or Ewens distributions)
- a more recent approach: look for a limit theorem for the permutation itself (interesting for non-uniform models or constrained permutations).

Introduction

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  (usually for uniform or Ewens distributions)

- a more recent approach: look for a limit theorem for the permutation itself (interesting for non-uniform models or constrained permutations).

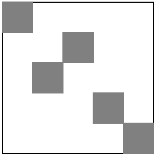
Today: present the theory of permutons and illustrate it with some results in the literature.

The theory of permutons
(Hoppen, Kohayakawa, Moreira, Rath, Sampaio)

How to look at large permutations?

A permutation π can be encoded as a probability measure μ_π on $[0,1]^2$.

$$\pi = 5\,3\,4\,2\,1 \quad \mapsto \quad \mu_\pi =$$



Each square has weight $1/n$ (i.e. density n).

We have a natural notion of limit for such objects: the weak convergence.
This defines a nice compact Polish space.

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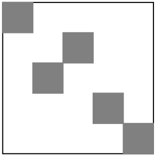

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Note that μ_π is a coupling of two uniform measures (in other words, has uniform marginals).

$\rightarrow$ potential limits also have uniform marginals.

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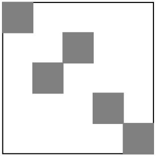

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Definition

A permuton is a probability measure on $[0,1]^2$ with uniform marginals.

Next few slides: connection with permutation patterns.

Permutation patterns

Definition

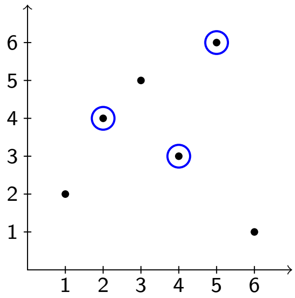
An occurrence of a pattern τ in σ is a subsequence $\sigma_{i_1} \dots \sigma_{i_k}$ that is order-isomorphic to τ, *i.e.* $\sigma_{i_s} < \sigma_{i_t} \;\Leftrightarrow\; \tau_s < \tau_t$.

Example (occurrences of 2 1 3)

2 4 5 3 6 1

8 2 3 4 6 1 7 5

Visual interpretation



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- Families of permutations avoiding given patterns (called permutation classes) appear in various domains: sorting algorithms, enumerative geometry, genomics.
- They are widely studied from an enumerative, algorithmic and more recently probabilistic point of view.
- Here we are more interested in numbers of occurrences of τ in σ.

Pattern density in permutations and permutons

If τ and σ are permutations of size k and n, resp., we set

$$\widetilde{\mathrm{occ}}(\tau,\sigma) := \binom{n}{k}^{-1} \cdot \# \left\{ \begin{array}{c} \text{occurrences of} \\ \tau \text{ in } \sigma \end{array} \right\} \in [0,1].$$

In other terms: take k elements uniformly at random in σ, the probability to find a pattern τ is $\widetilde{\mathrm{occ}}(\tau,\sigma)$.

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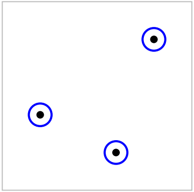
This probabilistic interpretation extends to permutons: replacing σ with a permuton μ

$$\widetilde{\mathrm{occ}}(\tau,\mu) := \mathbb{P}^{\mu}(U_1, \cdots, U_k \text{ form a pattern } \tau),$$

where $U_1, \cdots, U_k$ are i.i.d. points with distribution μ.



a 213 pattern

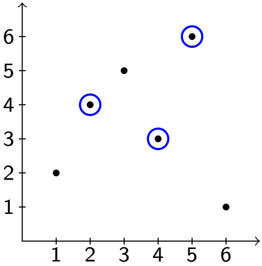
An approximation lemma

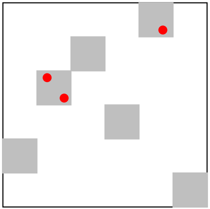
Reminder:

$$\widetilde{\mathrm{occ}}(\tau,\sigma) := \binom{n}{k}^{-1} \cdot \# \left\{ \begin{array}{c} \text{occurrences of} \\ \tau \text{ in } \sigma \end{array} \right\} \in [0,1].$$

$$\widetilde{\mathrm{occ}}(\tau,\mu) := \mathbb{P}^{\mu}(U_1, \cdots, U_k \text{ form a pattern } \tau),$$

⚠ In general, $\widetilde{\mathrm{occ}}(\tau,\sigma) \neq \widetilde{\mathrm{occ}}(\tau,\mu_\sigma)$.





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But we have the following approximation lemma:

Lemma

If π and σ are permutations of size k and n, resp., then

$$|\,\widetilde{\mathrm{occ}}(\pi,\sigma) - \widetilde{\mathrm{occ}}(\pi,\mu_\sigma)| \leq \frac{1}{n}\binom{k}{2}.$$

Pattern density convergence and permuton convergence

Theorem (Hoppen, Kohayakawa, Moreira, Rath, Sampaio, 2013)

Weak convergence of permutons is equivalent to the pointwise convergence of $\widetilde{\mathrm{occ}}(\tau, \cdot)$ for all τ, i.e.

$$\mu^{(n)} \to \mu \;\Leftrightarrow\; \textit{for all } \tau,\; \widetilde{\mathrm{occ}}(\tau, \mu^{(n)}) \to \widetilde{\mathrm{occ}}(\tau, \mu).$$

As a consequence, for a sequence of permutation $\sigma^{(n)}$ of size tending to infinity,

$$\mu_{\sigma^{(n)}} \to \mu \;\Leftrightarrow\; \text{for all } \tau,\; \widetilde{\mathrm{occ}}(\tau, \sigma^{(n)}) \to \widetilde{\mathrm{occ}}(\tau, \mu).$$

(In terms of permutations, $\widetilde{\mathrm{occ}}(\tau, \sigma^{(n)})$ is much more concrete!)

Proof that $\mu^{(n)} \to \mu \;\Rightarrow\; \forall \tau, \widetilde{\mathrm{occ}}(\tau, \mu^{(n)}) \to \widetilde{\mathrm{occ}}(\tau, \mu)$

Observe that

$$\begin{aligned}\widetilde{\mathrm{occ}}(\tau, \mu) &:= \mathbb{P}^{\mu}(U_1, \cdots, U_k \text{ form a pattern } \tau) \\ &= \int_{([0,1]^2)^k} \mathbf{1}\left[u_1, \cdots, u_k \text{ form a pattern } \tau\right] d\mu^{\otimes k}(u_1, \cdots, u_k)\end{aligned}$$

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If $\mu^{(n)} \to \mu$, then $(\mu^{(n)})^{\otimes k} \to \mu^{\otimes k}$ and the statement would be immediate if $(u_1, \cdots, u_k) \mapsto \mathbf{1}\big[u_1, \cdots, u_k \text{ form a pattern } \tau\big]$ was continuous.

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Its discontinuity set of $(u_1, \cdots, u_k) \mapsto \mathbf{1}\big[u_1, \cdots, u_k \text{ form a pattern } \tau\big]$ corresponds to k-uples where (at least) two u_i have one of their coordinates equal.

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But since μ has uniform marginals, this set has $\mu^{\otimes k}$ measure 0. This ends the proof.

Proof that $\forall \tau, \widetilde{\mathrm{occ}}(\tau, \mu^{(n)}) \to \widetilde{\mathrm{occ}}(\tau, \mu) \;\Rightarrow\; \mu^{(n)} \to \mu$

Claim: for any p, q there exists constants $c^{\tau}_{p,q}$ such that for all permutons μ,

$$\int_{[0,1]^2} x^p\, y^q\, d\mu(x,y) = \sum_{\tau} c^{\tau}_{p,q}\, \widetilde{\mathrm{occ}}(\tau, \mu).$$

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If the claim holds, then convergence of all $\widetilde{\mathrm{occ}}(\tau, \cdot)$ implies moment convergence, which in turn implies convergence in distribution.

So we only have to prove the claim.

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Consider U, $V_1, \dots,$ V_p, $W_1, \dots, W_q$ i.i.d points with distribution μ and the probability $\mathbb{P}\big[\forall i,\; x(V_i) \le x(U) \wedge y(W_i) \le y(U)\big]$.

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On the one hand, conditioning on $U = (x, y)$, we get

$$\begin{aligned}\mathbb{P}\big[\forall i,\ x(V_i) \le x(U) \wedge y(W_i) \le y(U)\big] \\ &= \int_{[0,1]^2} \mathbb{P}\big[\forall i,\ x(V_i) \le x \wedge y(W_i) \le y | U\big]\, d\mu(x,y) \\ &= \int_{[0,1]^2} x^p\, y^q\, d\mu(x,y).\end{aligned}$$

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On the other hand, the event $\{\forall i,\ x(V_i) \le x(U) \wedge y(W_i) \le y(U)\}$ can be written as a huge case disjonction, specifying all the order relations between x-coordinates and y-coordinates respectively (i.e. specifying the pattern formed by U, $V_1, \dots, V_p$, $W_1, \dots, W_q$).

Therefore $\mathbb{P}\big[\forall i,\ x(V_i) \le x(U) \wedge y(W_i) \le y(U)\big]$ is a linear combination of $\widetilde{\text{occ}}(\tau, \mu)$.

Convergent sequence of permutations

We proved:

Theorem (Hoppen, Kohayakawa, Moreira, Rath, Sampaio, 2013)

Weak convergence of permutons is equivalent to the pointwise convergence of $\widetilde{\mathrm{occ}}(\tau, \cdot)$ *for all* τ, i.e.

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Corollary

Let $\sigma^{(n)}$ *be a sequence of permutation such that* $\widetilde{\mathrm{occ}}(\tau, \sigma^{(n)})$ *converges for all* τ. *Then there exists a permuton* μ *such that* $\sigma^{(n)} \to \mu$.

Exercise: prove the corollary (hint: use compactness).

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Q: is every permuton μ the limit of some sequence of permutation?
YES, we will see a random construction in the next slide.

Random permutation model associated with a permuton

Fix a permuton μ.

We define a random permutation $\boldsymbol{\sigma}_n$ of size n as the pattern formed by n i.i.d random points $U_1, \cdots, U_n$ with distribution μ.

Example: if μ is the uniform measure on $[0,1]^2$, then $\boldsymbol{\sigma}_n$ is a uniform random permutation of size n.

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For a fixed τ,

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It is a (normalized) sum of variables with a sparse dependency graphs.

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Thus $\boldsymbol{\sigma}_n$ converges almost surely to μ, which proves the existence of sequences of permutations converging to μ.

Summary

1. Weak convergence of permutons is equivalent to convergence of pattern densities;
2. If the pattern densities of a sequence of permutation converge, then there exists a limit permuton.
3. All permutons are limits of some permutation sequence.

$\rightarrow$ the space of permutons is the natural space of limiting objects for the pattern density convergence.

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2 is very useful: we do not need to construct the limiting permuton, we know it exists. Is there an analogue for random permutations?

Permuton convergence of random permutations

Theorem (BBFGMP, 2017+)

Let $\boldsymbol{\sigma}_n$ be a random permutation of size n. The following assertions are equivalent.

(a) $\mu_{\boldsymbol{\sigma}_n}$ *converges in distribution for the weak topology to some random permuton $\boldsymbol{\mu}$.*

(b) *The random infinite vector* $\big(\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)\big)_{\pi \in \mathfrak{S}}$ *converges in distribution in the product topology to some random infinite vector* $(\boldsymbol{\Lambda}_\pi)_{\pi \in \mathfrak{S}}$.

(c) *For every π in $\mathfrak{S}$, there is a $\Delta_\pi \geq 0$ such that*

$$\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)] \xrightarrow{n \to \infty} \Delta_\pi .$$

Note: $(a) \Leftrightarrow (b)$ expected (random version of the previous result),
$(b) \Leftrightarrow (c)$ might be more surprising (cv in expectation is enough!).

Why are expectations enough?

Claim: Fix $\tau_1, \dots, \tau_k$. There exist constants c_ρ such that, for all permutons μ,

$$\prod_{i=1}^{k} \widetilde{\mathrm{occ}}(\tau_i, \mu) = \sum_{\rho} c_\rho \, \widetilde{\mathrm{occ}}(\rho, \mu).$$

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If the expectations converge, then the joint moments converge and we have (multi-variate) convergence in distribution.

Analogy with graphons

Permutons are inspired from recent developments in random graph theory

Permutations	Graphs
Pattern densities	Subgraph densities

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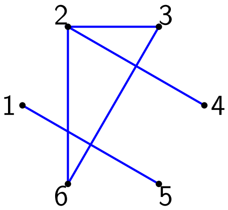
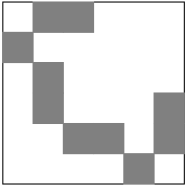
Measure on $[0,1]^2$	Functions $[0,1]^2 \to [0,1]$



$\mapsto$



(value 1 on gray rectangles)

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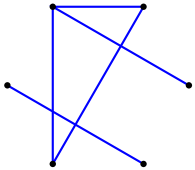
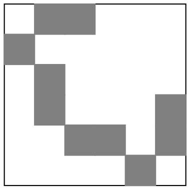
Permutations	Graphs
Pattern densities	Subgraph densities
Measure on $[0,1]^2$	Functions $[0,1]^2 \to [0,1]$
	up to composition with a Lebesgue-preserving isomorphism



$\mapsto$

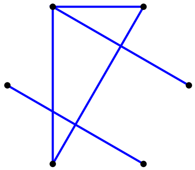


(value 1 on gray rectangles)

Analogy with graphons

Permutons are inspired from recent developments in random graph theory

Permutations	Graphs
Pattern densities	Subgraph densities
Measure on $[0,1]^2$	Functions $[0,1]^2 \to [0,1]$ up to composition with a Lebesgue-preserving isomorphism



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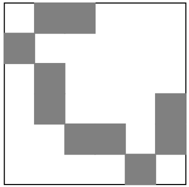


(value 1 on gray rectangles)

- cv of subgraph densities $\Leftrightarrow$ cv of graphon
- space of graphons is compact.

Some results/conjecture in the permuton framework

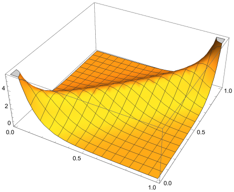
Limiting permuton for Mallows permutation (Starr, '09)

Mallows model on S_n: $\mathbb{P}(\sigma) \propto q^{\mathrm{inv}(\sigma)}$.

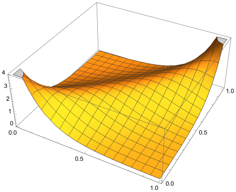
Theorem (Starr, '09)

Let $\sigma^{(n)}$ be a random permutation taken with the Mallows measure of parameter $q_n = 1 - \beta/n$. Then $\mu_{\sigma^{(n)}}$ converge to the deterministic permuton with density

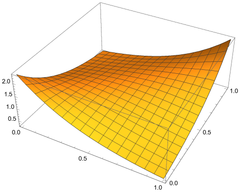
$$u(x,y) = \frac{(\beta/2)\sinh(\beta/2)}{\left(e^{\beta/4}\cosh(\beta[x-y]/2) - e^{-\beta/4}\cosh(\beta[x+y-1]/2)\right)^2}.$$



$\beta = 10$



$\beta = 6$



$\beta = 2$

A large deviation principle (Kenyon, Král, Radin, Winkler, '15)

Definition (entropy of a permuton μ with density g)

$$H(\mu) = \int_{[0,1]^2} -g(x,y) \log g(x,y) dxdy.$$

If μ has no density, $H(\mu) := \infty$.

Theorem (Trashorras, '08, KKRW, '15)

Let Λ be a set of permutons, Λ_n the set of permutations $\pi \in S_n$ with $\mu_\pi \in \Lambda$. Then:

1. *If Λ is closed,* $\limsup_{n\to\infty} \frac{1}{n} \log \frac{|\Lambda_n|}{n!} \leq \sup_{\mu\in\Lambda} H(\mu)$;
2. *If Λ is open,* $\liminf_{n\to\infty} \frac{1}{n} \log \frac{|\Lambda_n|}{n!} \geq \sup_{\mu\in\Lambda} H(\mu)$.

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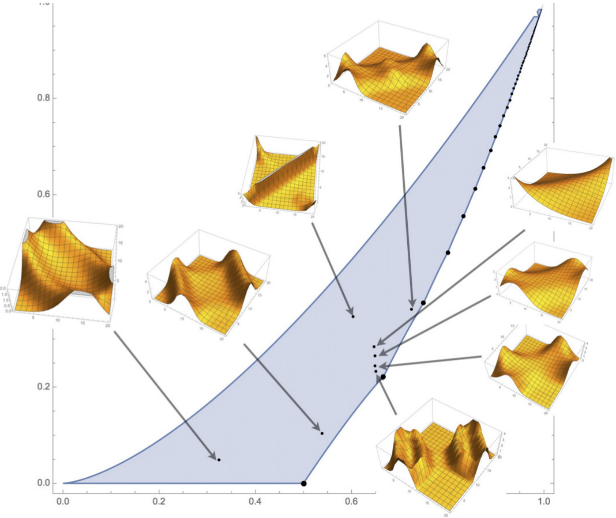
Q: which permutons maximize the entropy under some constraints? (such as fixing some pattern densities)

A nice picture (Kenyon, Král, Radin, Winkler, ’15)



x-axis: $\widetilde{\mathrm{occ}}(12, \mu)$
y-axis: $\widetilde{\mathrm{occ}}(123, \mu)$

blue zone: zone where there exists a permuton μ with such pattern densities.

Displayed permutons are entropy maximizers for fixed 12 and 123 densities.

©KKRW, ’15

Random minimal factorizations (Angel, Holroyd, Romik, Virag, '06)

Consider a uniform random minimal factorization of $\omega_0 := n$ n-1 ... 2 1 into transposition: $\omega_0 = \tau_1 \dots \tau_N$ (where $N = \binom{n}{2}$).
Q: what do partial porducts $\tau_1 \dots \tau_{\lfloor cN \rfloor}$ look like?

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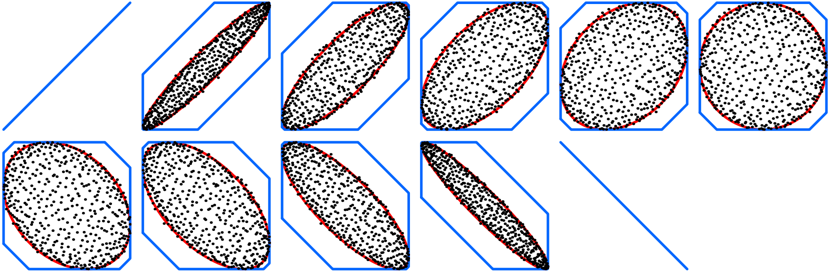
Pictures (©AHRV) ($n = 500, \; c = 0, .1, .2, \dots, .9, 1$):



There is a conjectural formula for the limiting process in the space of permutons.

Uniform random permutations in classes

Reminder: a class of permutations is the set of permutations that avoid given patterns. Notation: $\mathrm{Av}(\tau_1, \dots, \tau_r)$.

Recently studied from probabilistic point of view (Bevan, Hoffman, Madras, Pak, Rizzolo, Svilken, ...).

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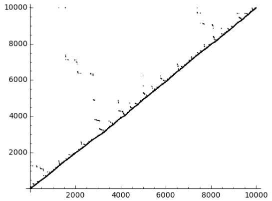
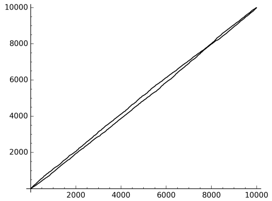
Start with simple classes $\mathrm{Av}(\tau)$ with $\tau \in S_3$. Not interesting from a permuton viewpoint: a uniform permutation in $\mathrm{Av}(\tau)$ with $\tau \in S_3$ concentrates on one of the diagonal (pictures ©Hoffman, Rizzolo, Svilken).



Random 231-avoiding permutation



Random 321-avoiding permutation

Limit of separable permutations (BBFGP, '16)

The class of separable permutations is $\mathrm{Av}(3142, 2413)$ (there are equivalent more natural definitions).

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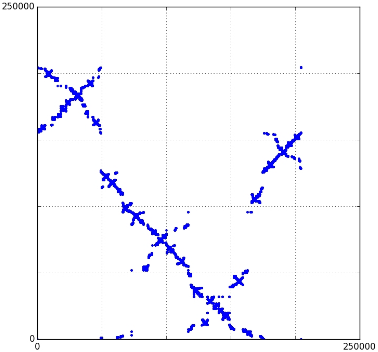
Let $\boldsymbol{\sigma}^{(n)}$ be a uniform random separable permutation of size n.

Theorem (BBFGP,'16)

$\mu_{\boldsymbol{\sigma}^{(n)}}$ tends towards a non-deterministic permuton $\boldsymbol{\mu}$, linked to the Brownian excursion

(Simulation on the right)



(More on that tomorrow)