Mod-ϕ convergence I: examples and probabilistic estimates

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Content of the four lectures

Two (mostly independent) parts

Lectures 1 and 2: mod-ϕ convergence, probabilistic estimates and dependency graphs;

Lectures 3 and 4: some recent work on random permutations: permutons, uniform permutations in classes.

Central limit theorem (CLT) and beyond

- Standard CLT: renormalized sum of i.i.d. variables with finite variance tends towards a Gaussian distribution.
- Many relaxation of the i.i.d. hypothesis: CLT for Markov chains, martingales, mixing processes, m-dependent sequence, "associated" random variables...

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- Many relaxation of the i.i.d. hypothesis: CLT for Markov chains, martingales, mixing processes, m-dependent sequence, "associated" random variables...
- We often have *companion theorems*: deviation principles, concentration inequalities, local limit theorem, speed of convergence...

But the companion theorems need extra effort to prove.
Philosophy: Mod-ϕ is a universality class beyond the CLT, which implies some companion theorems.

Mod-ϕ convergence: definition

Setting:

- D a domain of $\mathbb{C}$ containing 0.
- ϕ infinite divisible distribution with Laplace transform $\exp(\eta(z))$ on D.

Definition (Nikeghbali, Kowalski)

A sequence of real r.v. (X_n) converges mod-ϕ on D with parameter $t_n \to \infty$ and limiting function ψ if, locally uniformly on D,

$$\exp(-t_n\,\eta(z))\,\mathbb{E}(\mathrm{e}^{zX_n}) \to \psi(z), \tag{1}$$

Informal interpretation:

- $X_n = t_n$ independent copies of ϕ + perturbation encoded in ψ.
- instead of renormalizing the variables as in CLT, we renormalized the Fourier/Laplace transform to get access to the next term.

(this notion has some similarity with Hwang's quasi-powers.)

Mod-ϕ convergence implies a CLT

Proposition

If (X_n) converges mod-ϕ on D with parameter t_n, then

$$Y_n = \frac{X_n - t_n \eta'(0)}{\sqrt{t_n \eta''(0)}} \longrightarrow_d \mathcal{N}(0,1).$$

Proof: easy, use the mod-ϕ estimate to show that $\mathbb{E}(\mathrm{e}^{\zeta Y_n})$ converges pointwise to $\mathrm{e}^{\zeta^2/2}$.

Philosophy: Many classical ways of proving CLTs can be adapted to prove mod-ϕ convergence.
(In particular, in all examples in the next few slides, the CLT is a well-known result.)

Outline of today's talk

1. Introduction: CLT and mod-ϕ convergence
2. Examples of mod-ϕ convergence sequences
   - How to prove mod-ϕ convergence
3. Companion theorems
   - Speed of convergence
   - Deviation and normality zone

Examples with an explicit generating function (1/3)

We start with a trivial example.

Let $Y_1, Y_2, \dots$ be i.i.d. with law ϕ and W_n a sequence of r.v., independent from the Y, whose Laplace transform converges to that of W on D.
Set $X_n = W_n + \sum_{i=1}^n Y_i$. Then

$$\mathbb{E}(\mathrm{e}^{zX_n}) = \mathrm{e}^{n\,\eta(z)}\,\mathbb{E}(\mathrm{e}^{zW_n}) = \mathrm{e}^{n\,\eta(z)}\,(\mathbb{E}(\mathrm{e}^{zW}) + o(1)).$$

Thus X_n converges mod-ϕ with parameters $t_n = n$ and limiting function $\psi(z) = \mathbb{E}(\mathrm{e}^{zW})$.

Examples with an explicit generating function (2/3)

Let X_n be the number of cycles in a uniform random permutation.

$$\mathbb{E}[\mathrm{e}^{zX_n}] = \prod_{i=1}^{n} \left(1 + \frac{\mathrm{e}^z - 1}{i}\right) = \mathrm{e}^{H_n(\mathrm{e}^z - 1)} \prod_{i=1}^{n} \frac{1 + \frac{\mathrm{e}^z - 1}{i}}{\mathrm{e}^{\frac{\mathrm{e}^z - 1}{i}}}.$$

where $H_n = \sum_{i=1}^{n} \frac{1}{i} = \log n + \gamma + \mathcal{O}(n^{-1})$.

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where $H_n = \sum_{i=1}^n \frac{1}{i} = \log n + \gamma + \mathcal{O}(n^{-1})$. The product on the right-hand side converges locally uniformly on $\mathbb{C}$ to an infinite product, which turns out to be related to the Γ function,

$$\mathbb{E}[\mathrm{e}^{zX_n}]\,\mathrm{e}^{-(\mathrm{e}^z-1)\log n} \to \mathrm{e}^{\gamma\,(\mathrm{e}^z-1)} \prod_{i=1}^{\infty} \frac{1 + \frac{\mathrm{e}^z-1}{i}}{\mathrm{e}^{\frac{\mathrm{e}^z-1}{i}}} = \frac{1}{\Gamma(\mathrm{e}^z)}$$

locally uniformly, *i.e.*, one has mod-Poisson convergence on $\mathbb{C}$ with parameters $t_n = \log n$ and limiting function $1/\Gamma(\mathrm{e}^z)$.

Examples with an explicit generating function (3/3)

Other examples with explicit generating functions:

- $\log(|\det(\mathrm{Id} - U_n)|)$ where U_n is an unitary Haar-distributed random matrices. It converges mod-Gaussian on {Re(z)>-1} with parameter $\frac{\log n}{2}$ and limiting function $\Psi_1(z) = \frac{G(1+z/2)^2}{G(1+z)}$ (G is the G-Barnes function).

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- Let M_n be a GUE matrix. Then $|\det(M_n)| - \mathbb{E}(|\det(M_n)|)$ converges mod-Gaussian on $\{|z| < 1\}$ with parameter $t_n \sim \frac{1}{2}\log(n)$ and same limiting function $\Psi_1(z)$ (Döring, Eichelsbacher, 2013).

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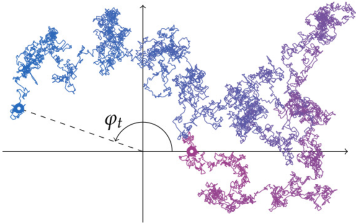
- φ_t is the winding number of a Brownian motion starting at 1. It "converges mod-Cauchy on $i\mathbb{R}$" with parameter $\frac{\log(8t)}{2}$ and limiting function $\Psi_2(i\zeta) = \frac{\sqrt{\pi}}{\Gamma((|\zeta|+1)/2)}$.



Examples with an explicit **bivariate** generating function (overview)

Number $\omega(k)$ of prime divisors of the integer k

$$\sum_{k \geq 1} \frac{\mathrm{e}^{z\omega(k)}}{k^s} = \prod_p \left(1 + \frac{\mathrm{e}^z}{p^s(1-p^{-s})}\right).$$

$\Omega_n = \omega(k)$, for a uniform random positive integer $k \leq n$.

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Number of ascents A_n in a random permutation of size n

$$\sum_{n \geq 1} \mathbb{E}(\mathrm{e}^{zA_n}) t^n = \frac{\mathrm{e}^z - 1}{\mathrm{e}^z - \mathrm{e}^{t(e^z-1)}}.$$

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Number $\omega(k)$ of prime divisors of the integer k

$$\sum_{k \geq 1} \frac{\mathrm{e}^{z\omega(k)}}{k^s} = \prod_p \left(1 + \frac{\mathrm{e}^z}{p^s(1-p^{-s})}\right). \quad \Omega_n \text{ converges mod-Poisson}$$

$\Omega_n = \omega(k)$, for a uniform random positive integer $k \leq n$.

Number of ascents A_n in a random permutation of size n

$$\sum_{n \geq 1} \mathbb{E}(\mathrm{e}^{zA_n}) t^n = \frac{\mathrm{e}^z - 1}{\mathrm{e}^z - \mathrm{e}^{t(e^z-1)}}. \quad A_n \text{ "converges mod-}U([0,1])\text{"}$$

In both cases one can extract the Laplace transform of Ω_n or A_n by a path integral and study asymptotics.

A central limit theorem due to Harper

Theorem (Harper, 1967)

Let X_n be a $\mathbb{N}$-valued random variable such that $P_n(t) = \mathbb{E}(t^{X_n})$ has nonpositive real roots. Assume $\mathrm{Var}(X_n) \to \infty$. *Then*

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathrm{Var}(X_n)}} \longrightarrow_d \mathcal{N}(0,1).$$

Example: X_n is the number of blocks of a uniform random set-partitions. (One can prove that

$$P_n(t)e^t = \mathrm{cst}_n t \frac{d}{dt}(P_{n-1}(t)\mathrm{e}^t)$$

and apply Rolle's theorem inductively.)

Mod-Gaussian convergence in Harper's theorem

Theorem (FMN, 2016)

Let X_n be a $\mathbb{N}$-valued random variable such that $P_n(t) = \mathbb{E}(t^{X_n})$ is a polynomial with nonpositive real roots. Denote $\sigma_n^2 = \mathrm{Var}(X_n)$ and $L_n^3 = \kappa_3(X_n)$ the second and third cumulants of X_n and assume $1 \ll L_n \ll \sigma_n \ll L_n^2$.
Then $\frac{X_n - \mathbb{E}(X_n)}{L_n}$ converges mod-Gaussian on $\mathbb{C}$ with parameters $t_n = \frac{\sigma_n^2}{L_n^2}$ and limiting function $\psi = \exp(z^3/6)$.

Idea of proof: X_n write as a sum of N_n Bernoulli variables B_k (of unknown parameters). Thus

$$\mathbb{E}(\mathrm{e}^{zX_n}) = \prod_{k=1}^{N_n} \mathbb{E}(\mathrm{e}^{zB_k})$$

and we do Taylor expansions on the right-hand side.

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Example: X_n is the number of blocks of a uniform random set-partitions. (The third cumulant estimate is not trivial.)

Adapting the method of moments (1/3)

Instead of moments we use cumulants $\kappa_r(X_n)$. If X is a random variable, its cumulants are the coefficients of

$$\log \mathbb{E}[e^{zX}] = \sum_{r=1}^{\infty} \frac{\kappa^{(r)}(X)}{r!} z^r.$$

First cumulants:

$$\begin{aligned}
\kappa_1(X) &:= \mathbb{E}(X), \\
\kappa_2(X) &:= \mathrm{Var}(X, Y) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
\kappa_3(X) &:= \mathbb{E}(X^3) - 3\mathbb{E}(X^2)\mathbb{E}(X) + 2\mathbb{E}(X)^3.
\end{aligned}$$

Fact: Y_n converge in distribution to $\mathcal{N}(0,1)$ if $Var(Y_n) \to 1$ and all other cumulants tend to 0.

Adapting the method of moments (2/3)

Definition (uniform control on cumulants)

A sequence (S_n) admits a uniform control on cumulants with DNA (D_n, N_n, A) and limits σ^2 and L if $D_n = o(N_n)$, $N_n \to +\infty$ and

$$\forall r \geq 2, \;\; |\kappa^{(r)}(S_n)| \leq N_n\,(2D_n)^{r-1}\,r^{r-2}\,A^r;$$

$$\frac{\kappa^{(2)}(S_n)}{N_n\,D_n} = (\sigma_n)^2 \to_{n\to\infty} \sigma^2; \qquad \frac{\kappa^{(3)}(S_n)}{N_n\,(D_n)^2} = L_n \to_{n\to\infty} L.$$

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Proposition

Take (S_n) admits a uniform control on cumulants with $\sigma^2 > 0$. Then, $X_n := \dfrac{S_n - \mathbb{E}[S_n]}{(N_n)^{\frac{1}{3}}(D_n)^{\frac{2}{3}}}$ *converges mod-Gaussian on $\mathbb{C}$, with $t_n = (\sigma_n)^2\left(\frac{N_n}{D_n}\right)^{\frac{1}{3}}$ and limiting function $\psi(z) = \exp(\frac{Lz^3}{6})$.*

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Remark

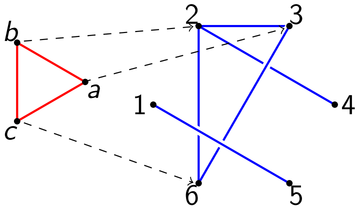
Uniform bounds on cumulants have been studied (in more generality) by Saulis and Statulevičius (1991) (see Döring-Eichelsbacher 2012, 2013, for numerous applications).

In this context, we don't have new theoretical results, but new examples.

Adapting the method of moments (3/3): a new example

If $F = (V_F, E_F)$ and $G = (V_G, E_G)$ are finite graphs, a copy of F in G is a map $\psi : V_F \to V_G$ such that

$$\forall e = \{x, y\} \in E_F, \ \{\psi(x), \psi(y)\} \in E_G.$$



Proposition

The number of copies of a fixed F in $G(n,p)$ (p fixed) admits a uniform control on cumulants with DNA $(n^{|V_G|-2}, n^{|V_G|}, 1)$ and $\sigma^2 > 0$.

(behind this: dependency graphs, more on that and more examples tomorrow!)

Transition

Reminder: if X_n converges mod-ϕ, then $Y_n = \frac{X_n - t_n \eta'(0)}{\sqrt{t_n \eta''(0)}}$ converges to a standard Gaussian, i.e., for a fixed y,

$$\lim_{n\to\infty} \mathbb{P}\big(Y_n \geq y\big) = \frac{1}{\sqrt{2\pi}} \int_y^\infty \mathrm{e}^{-u^2/2} du =: F_{\mathcal{N}}(y). \qquad \text{(CLT)}$$

Main questions

Speed of convergence What is the error term (uniformly in y) in (CLT)?

Deviation probability What if $y \to \infty$? The limit is 0 but can we give an equivalent?

A first bound for the speed of convergence

Proposition (FMN, 2016)

Let X_n converges mod-ϕ on a domain D containing $i\mathbb{R}$. Assume ϕ non-lattice. Then

$$\mathbb{P}\big(Y_n \geq y\big) = F_{\mathcal{N}}(y) + \frac{\psi'(0)}{\sqrt{t_n \eta''(0)}} F_1(y) + \frac{\eta'''(0)}{6\sqrt{t_n (\eta''(0))^3}} F_2(y) + o(\tfrac{1}{\sqrt{n}}),$$

for explicit functions $F_1(y)$ and $F_2(y)$ (Gaussian integrals).
In particular, the error term in (CLT) is $\mathcal{O}\big(t_n^{-1/2}\big)$ and it is optimal if $\psi'(0) \neq 0$ or $\eta'''(0) \neq 0$.

$\rightarrow$ Tight bound for $\log(\det(\mathrm{Id} - U_n))$, but not for triangle count (see later)...

Bound on speed of convergence: ideas of proof

(Close to Feller, 1971, for the i.i.d. case.)

Standard tool in this context: Berry's inequality for centered variables

$$|F(y) - G(y)| \leq \frac{1}{\pi} \int_{-T}^{T} \left| \frac{f^*(\zeta) - g^*(\zeta)}{\zeta} \right| d\zeta + \frac{24m}{\pi T}.$$

F and G are distribution functions; f^* and g^* the Fourier transform of the corresponding laws; m a bound on the density g.

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Take $F_n(y) = \mathbb{P}\big(Y_n \geq y\big)$ and

$$G_n(y) = \int_{-\infty}^{y} \left(1 + \frac{\psi'(0)}{\sqrt{t_n \eta''(0)}}\, u + \frac{\eta'''(0)}{6\sqrt{t_n (\eta''(0))^3}}\, (u^3 - 3u) \right) g(u)\, du.$$

The mod-ϕ estimate allows you to control the integral for $T = \Delta t_n^{1/2}$.
(For $\zeta \ll t_n^{1/2}$, $f^*(\zeta) \sim g^*(\zeta)$, for $\zeta \approx t_n^{1/2}$, both terms are small.)

Make n tends to infinity and then Δ.

Speed of convergence for triangles in random graphs

Let T_n be the number of copies of $F = K_3$ in $G(n, p)$.

- Our bound gives an error term $\mathcal{O}(n^{-1/3})$.
- With a result of Rinott (1994), we can get $O(n^{-1})$ (see also Krokowski, Reichenbachs and Thaele, 2015).

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Question

Can we improve our bounds in this case?

A better bound from uniform control on cumulants

Proposition (Saulis, Statelivičius, 1991, FMN, 2017)

Let (S_n) be a sequence with a uniform control on cumulants with DNA (D_n, N_n, A) with $\sigma^2 > 0$.
(In particular, $|\kappa^{(r)}(S_n)| \leq N_n\,(2D_n)^{r-1}\,r^{r-2}\,A^r$.)
Then the error term in (CLT) is $\mathcal{O}(t_n^{-3/2}) = \mathcal{O}(\sqrt{D_n/N_n})$.

In case of triangles, we get $\mathcal{O}(n^{-1})$ as Rinott (1994) or Krokowski, Reichenbachs and Thaele (2015).

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Proof: again Berry's inequality

$$|F(x) - G(x)| \leq \frac{1}{\pi} \int_{-T}^{T} \left| \frac{f^*(\zeta) - g^*(\zeta)}{\zeta} \right| d\zeta + \frac{24m}{\pi T}.$$

but we have a better control on $f^*(\zeta)$ and thus we can choose $T = t_n^{3/2}$.

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In case of triangles, we get $\mathcal{O}(n^{-1})$ as Rinott (1994) or Krokowski, Reichenbachs and Thaele (2015).

Our statement is a bit more general: holds in the context of mod-stable convergence with additional control of the Laplace transform.

Example: winding number φ_t of a Brownian motion converges to a Cauchy law after renormalization at speed $\mathcal{O}(t_n^{-1}) = \mathcal{O}((\log\, n)^{-1})$.

Deviation probability

Theorem (FMN, 2016)

Assume X_n converges mod-ϕ (ϕ non-lattice) on a strip $\{|Re(z)| \leq C\}$. Let x_n bounded by C with $x_n \gg t_n^{-1/2}$. Then

$$\mathbb{P}\big(X_n - t_n\eta'(0) \geq t_n x_n\big) \sim_{n\to\infty} \frac{\exp(-t_n F(x_n))}{h_n\sqrt{2\pi t_n \eta''(h_n)}}\psi(h_n)(1+o(1)).$$

Here $F(x) = \sup_{h\in\mathbb{R}}(hx - \eta(h))$ is the Legendre Fenchel transform of η and h_n is the maximizer for $F(x_n)$.

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Standard proof strategy: applying speed of convergence result to the exponentially tilted variables $\widetilde{X}_n$:

$$\mathbb{P}[\widetilde{X}_n \in du] = \frac{\mathbb{E}^{hu}}{\varphi_{X_n}(h)}\,\mathbb{P}[X_n \in du].$$

$\widetilde{X}_n$ also converge mod-ϕ: its Laplace transform is simply $\mathbb{E}[\mathrm{e}^{z\,\widetilde{X}_n}] = \frac{\mathbb{E}[\mathrm{e}^{(z+h)\,X_n}]}{\mathbb{E}[\mathrm{e}^{h\,X_n}]}$.

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Similar result for lattice distributions ϕ: replace h in denominator by $e^h - 1$.

Normality zone

Definition

We say that Y_n has a normality zone $o(a_n)$ if (CLT) gives an equivalent of the tail probability for $y = o(a_n)$ but not for $y = O(a_n)$.

Proposition

Let X_n converges mod-ϕ on a strip $\{|Re(z)| \leq C\}$.
Then the normality zone of $\frac{X_n - t_n\eta'(0)}{\sqrt{t_n\eta''(0)}}$ is $o(t_n^{1/2-1/m})$, where $m \geq 3$ is minimal such that $\eta^{(m)} \neq 0$.
If ϕ is Gaussian, $m = \infty$ by convention, but we need to assume that $\psi \not\equiv 1$.

Additionally, we know the correction at the edge of the normality zone.

Some explicit results

- Let T_n be the number of copies of $F = K_3$ in $G(n,p)$. Then
$$\mathbb{P}\big[T_n \geq n^3 p^3 + n^2(v - 3p^3)\big] \sim \sqrt{\frac{9p^5(1-p)}{\pi\, v^2}}\, \exp\left(-\frac{v^2}{36\, p^5(1-p)} + \frac{(7-8p)\, v^3}{324\, n\, p^8(1-p)^2}\right)$$
for $1 \ll v = O(n^{2/3})$.

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for $1 \ll v = O(n^{2/3})$.
- Let A_n be the number of ascents in a random permutation of size n.
$$\mathbb{P}\left[A_n \geq \frac{n+1}{2} + \sqrt{\frac{n+1}{12}}\, y\right] = \frac{(1+o(1))}{y\sqrt{2\pi}}\, \exp\left(-\frac{y^2}{2} + \frac{y^4}{120(n+1)}\right)$$
for any positive y with $y = o(n^{\frac{5}{12}})$.

Some explicit results

- Let T_n be the number of copies of $F = K_3$ in $G(n,p)$. Then
$$\mathbb{P}\left[T_n \geq n^3p^3 + n^2(v-3p^3)\right] \sim \sqrt{\frac{9p^5(1-p)}{\pi\, v^2}}\, \exp\left(-\frac{v^2}{36\, p^5(1-p)} + \frac{(7-8p)\, v^3}{324\, n\, p^8(1-p)^2}\right)$$
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- Let U_n be Haar distributed in the unitary group $\mathrm{U}(n)$, one has: for $(\log n)^{-1/2} \ll x_n \ll 1$,
$$\mathbb{P}_n\left[\,|\det(\mathrm{Id} - U_n)| \geq n^{\frac{x_n}{2}}\right] = \frac{G(1+\frac{x_n}{2})^2}{G(1+x_n)}\, \frac{1}{x_n\, n^{\frac{x_n^2}{4}}\, \sqrt{\pi \log n}}\, (1+o(1)).$$

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(We also have estimates for negative deviations in all cases.)

Conclusion

Future work:

- Concentration estimates, local limit theorems. . .
- Prove mod-ϕ convergence in other contexts where the CLT is known: martingales, Stein exchangeable pairs, linear statistics of determinental processes, mixing processes . . .

Tomorrow: many examples of bounds on cumulants coming from dependency graphs, including subgraph counts/Ising model/symmetric simple exclusion process.