Asymptotics for skew standard Young tableaux via bounds for characters

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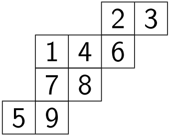
Outline of the talk

Denote $f^{\lambda/\mu}$ the number of standard Young tableaux of skew shape λ/μ.

General question: how does $f^{\lambda/\mu}$ behave when $|\lambda|, |\mu| \to \infty$?

Wide question: answer depends on the "shapes" of λ and μ (do they have large rows/columns? are they balanced?), the relative size of λ and μ.



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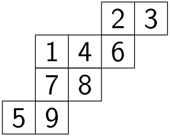

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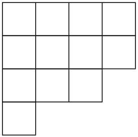
Our result: estimates/upper bound for *balanced* diagrams, when $|\mu| \ll |\lambda|$.

Main tool: bounds on symmetric group characters (F., Śniady, '11).

Basic definitions

- A partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n is a nonincreasing list of nonnegative integers of sum $|\lambda| = n$;
- It is identified with its Young diagram, formed by left-aligned row of boxes, with λ_1 boxes in the 1st row, λ_2 in the second, and so on...;



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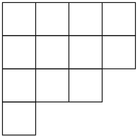
  Convention: $n := |\lambda|$, $k := |\mu|$



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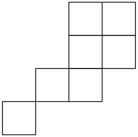

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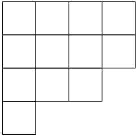
- A standard Young tableau (SYT) of (skew) shape λ/μ is a filling of λ/μ with integers from 1 to $|\lambda/\mu|$ with increasing rows and columns.
- $f^{\lambda/\mu}$ is the number of SYT of shape λ/μ.



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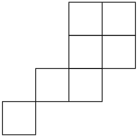

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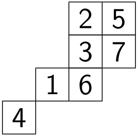

SYT of shape λ/μ

Straight tableaux

Theorem (Hook formula, Frame, Robinson, Thrall, '54)

For a straight shape λ,

$$f^{\lambda} = n! \prod_{\square \in \lambda} h(\square)^{-1}$$

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hook lengths

Asymptotics: Let λ be a diagram with at most $L\sqrt{n}$ rows and columns (called *balanced*). Most hook-lengths are of order $\Theta(\sqrt{n})$.

$$\log(f^{\lambda}) = \log(n!) - \tfrac{1}{2} n \log(n) - \sum_{\square \in \lambda} \log\left(\frac{h(\square)}{\sqrt{n}}\right)$$
$$= \tfrac{1}{2} n \log(n) + \mathcal{O}(n).$$

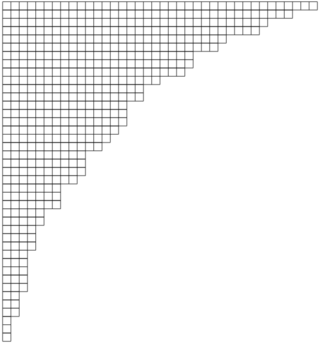
The $\mathcal{O}$ term can be written as an integral over the "limit shape" of λ.

Motivation from discrete probability theory

Plancherel measure on the set of Young diagrams of size n:

$$\mathbb{P}(\lambda) = \frac{(f^\lambda)^2}{n!}$$

(Vershik-Kerov, Logan-Shepp, '77) The limit shape is the one that maximizes the $\mathcal{O}(n)$ term in the previous slide.



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Fix a *straight* shape λ and consider a uniform standard tableau T of shape λ (Romik-Pittel, Biane, Śniady, Sun, …). Let $T^{(k)}$ be the diagram formed by boxes with entries at most k in T. Then

$$\mathbb{P}(T^{(k)} = \mu) = \frac{f^{\lambda/\mu} f^\mu}{f^\lambda}.$$

$\rightarrow$ we need the asymptotics of $f^{\lambda/\mu}$.

Asymptotics for $|f^{\lambda/\mu}|$: previous results

- (Kerov, Stanley independently): μ fixed, $\frac{\lambda_i}{n} \to \alpha_i$, $\frac{\lambda'_i}{n} \to \beta_i$,
$$f^{\lambda/\mu} \sim f^{\lambda} s_{\mu}(\boldsymbol{\alpha}| -\boldsymbol{\beta}|\gamma),$$
where $s_{\mu}(\boldsymbol{\alpha}| -\boldsymbol{\beta}|\gamma)$ is a super Schur function (definition later) and $\gamma = 1 - \sum \alpha_i - \sum \beta_i$.

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  In particular, when $\alpha_i = \beta_i = 0$ for all i (no rows or columns of size $\Theta(n)$), we have
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  (Consequence for a uniform random Young tableau T of shape λ:

$$\mathbb{P}(T^{(k)} = \mu) \sim \frac{(f^\mu)^2}{|\mu|!}.$$

  In words: fixed size truncations are asymptotically Plancherel distributed.)

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- (Morales-Pak-Panova-Tassy): various results for $k, n-k = \Theta(n)$, all of the form

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$\to$ we will consider intermediate ranges between μ fixed and $|\mu| = \Theta(\lambda)$.

Our results

For simplicity, we assume λ and μ balanced. We set $A_{\lambda/\mu} := k! \frac{f^{\lambda/\mu}}{f^\lambda f^\mu}$.

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Theorem (Dousse, F., '17)

1. *if* $k = o(n^{1/3})$, *then* $A_{\lambda/\mu} = 1 + \mathcal{O}\Big(k^{3/2} n^{-1/2}\Big)$.
2. *if* $k = o(n^{1/2})$, *then* $A_{\lambda/\mu} \leq \exp\Big[\mathcal{O}\Big(k^{3/2} n^{-1/2}\Big)\Big]$.
3. *if* $k = \omega(n^{1/2})$, *then* $A_{\lambda/\mu} \leq \exp\Big[k \log \frac{k^2}{n} + \mathcal{O}(k)\Big]$.

$R = \mathcal{O}(f(k,n))$ should be understood as follows: there exists a constant $C = C(L)$ such that $|R| \leq C\, f(|\mu|, n)$ for any λ and μ with at most $L\sqrt{n}$ (resp. $L\sqrt{|\mu|}$) rows and columns.

How to get asymptotics for $f^{\lambda/\mu}$?

- No multiplicative formula in general;
  For some family of skew-shapes, $f^{\lambda/\mu}$ admits a product formula
  $\rightarrow$ convenient to see if a bound is sharp/make conjectures, but not to prove bounds. . .

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- Recent "additive" hook formula for skew shapes (Naruse), used in this context by Morales-Pak-Panova-Tassy.
- We will use representation theory instead (as Kerov-Stanley).

Branching rule and $f^{\lambda/\mu}$

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$$\rho_\lambda \, / S_{n-1} \simeq \bigoplus_{\nu:\, \nu \nearrow \lambda} \rho_\nu .$$

$\nu \nearrow \lambda$ means $\nu \subseteq \lambda$ and $|\nu| = |\lambda| - 1$.

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Sequences $\mu = \nu^{(0)} \nearrow \cdots \nearrow \nu^{(r)} = \lambda$ correspond to SYT of shape λ/μ.

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Corollary (Stanley, '01): $f^{\lambda/\mu} = \frac{1}{k!} \sum_{\sigma \in S_k} \chi^\lambda(\sigma)\chi^\mu(\sigma)$.
χ^λ: character (=trace) of the representation ρ_λ.

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$\to$ use asymptotic results for character values to get asymptotics for $f^{\lambda/\mu}$.

Warm up: fixed k asymptotics (Kerov-Stanley)

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$$=: p_{\rho(\sigma)}(\boldsymbol{\alpha} | -\boldsymbol{\beta} | \gamma),$$

where $\rho(\sigma)$ is the cycle-type of σ
and $p_k(\boldsymbol{\alpha} | -\boldsymbol{\beta} | \gamma) := \sum_i \alpha_i^k - \sum_i (-\beta_i)^k + \delta_{k,1}\gamma$.

(In particular $p_1(\boldsymbol{\alpha} | -\boldsymbol{\beta} | \gamma) = 1$ since $\gamma := 1 - \sum_i \alpha_i - \sum_i \beta_i$.)

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Consequence:

$$\lim_{n\to\infty} \frac{f^{\lambda/\mu}}{f^\lambda} = \frac{1}{k!} \sum_{\sigma \in S_k} \chi^\mu(\sigma) p_{\rho(\sigma)}(\boldsymbol{\alpha}| -\boldsymbol{\beta}|\gamma) =: s_\mu(\boldsymbol{\alpha}| -\boldsymbol{\beta}|\gamma).$$

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- (Biane, '98) if λ is balanced,

$$\frac{\chi^\lambda(\sigma)}{f^\lambda} \sim C n^{-|\sigma|/2},$$

  where $|\sigma|$ is the number of transpositions needed to factorize σ (sometimes called absolute or reflection length).

  (If σ is in S_m, then $m - |\sigma|$ is the number of cycles of σ.)

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Consequence (the term $\sigma = \mathrm{id}$ dominates asymptotically):

$$\frac{f^{\lambda/\mu}}{f^\lambda} = \frac{f^\mu}{k!} + \mathcal{O}\left(n^{-1/2}\right)$$

Bounds on symmetric group characters

When k also grows to $+\infty$, we need bounds on characters on varying σ.

Theorem (F.-Śniady, '11)

There exists a constant $a > 1$, such that for every partition $\nu \vdash m$ and every permutation $\sigma \in S_m$,

$$\left| \frac{\chi^\nu(\sigma)}{f^\nu} \right| \leq \left[a \max\left(\frac{r(\nu)}{m}, \frac{c(\nu)}{m}, \frac{|\sigma|}{m} \right) \right]^{|\sigma|}.$$

$r(\nu)$, $c(\nu)$: numbers of rows and columns of ν, respectively.

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When ν is balanced, there are two regimes:

- if $|\sigma| = \mathcal{O}(\sqrt{m})$, then $\frac{\chi^\nu(\sigma)}{f^\nu} = \mathcal{O}\left(m^{-|\sigma|/2}\right)$;
- if $|\sigma| = \omega(\sqrt{m})$, then $\frac{\chi^\nu(\sigma)}{f^\nu} = \mathcal{O}\left((|\sigma|/m)^{-|\sigma|}\right)$.

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- For fixed $|\sigma|$, the bound is optimal up to a multiplicative constant.
- For large $|\sigma|$, it's very bad: LHS is known to be at most 1, while the RHS grows exponentially in m.

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and apply the previous bound on characters.

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- We have $|\sigma| \leq k = o(n^{1/2})$, so we always have $\left(\frac{\chi^\lambda(\sigma)}{f^\lambda}\right) = \mathcal{O}(n^{-|\sigma|/2})$;
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$$A_{\lambda/\mu} = S_1 + S_2 \text{ with } \begin{cases} |S_1| \leq \sum_{|\sigma| \leq \sqrt{k}} \left(\frac{aL}{\sqrt{n}}\right)^{|\sigma|} \left(\frac{aL}{\sqrt{k}}\right)^{|\sigma|}; \\ |S_2| \leq \sum_{|\sigma| > \sqrt{k}} \left(\frac{aL}{\sqrt{n}}\right)^{|\sigma|} \left(\frac{a|\sigma|}{k}\right)^{|\sigma|}. \end{cases}$$

Proof that $A_{\lambda/\mu} \le \exp\left[\mathcal{O}\left(k^{3/2} n^{-1/2}\right)\right]$ for $k = o(n^{1/2})$

We start from

$$A_{\lambda/\mu} = k! \frac{f^{\lambda/\mu}}{f^\lambda f^\mu} = \sum_{\sigma \in S_k} \left(\frac{\chi^\lambda(\sigma)}{f^\lambda} \right) \left(\frac{\chi^\mu(\sigma)}{f^\mu} \right)$$

and apply the previous bound on characters.

- We have $|\sigma| \le k = o(n^{1/2})$, so we always have $\left(\frac{\chi^\lambda(\sigma)}{f^\lambda} \right) = \mathcal{O}(n^{-|\sigma|/2})$;
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We need to control the number of σ with a given value of $|\sigma|$.

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Lemma (F., Śniady, '11)

For all $k, i \in \mathbb{N}$, we have

$$\#\{\sigma \in S_k : |\sigma| = i\} \leq \frac{k^{2i}}{i!}.$$

Proof:

$$\begin{aligned}\#\{\sigma \in S_k : |\sigma| = i\} &= [x^i](x+1)\cdots((k-1)x+1) \\ &\leq [x^i](kx+1)^k = \binom{k}{i} k^i \leq \frac{k^{2i}}{i!}. \quad \square\end{aligned}$$

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$$|S_1| \le \sum_{i=0}^{\sqrt{k}} \frac{k^{2i}}{i!} \left(\frac{aL}{\sqrt{n}}\right)^i \left(\frac{aL}{\sqrt{k}}\right)^i$$

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Similarly, S_2 can be bound by the sum of a convergent geometric series $\sum_{i>\sqrt{k}}(C'kn^{-1/2})^i \approx (C'kn^{-1/2})^{\sqrt{k}}$. □

Improving the bounds?

- We proved: when $k = o(n^{1/2})$,
$$A_{\lambda/\mu} \leq \exp\left[\mathcal{O}\left(k^{3/2} n^{-1/2}\right)\right].$$
Moreover, we can find families of shapes λ/μ with $k = n^{\alpha}$, (for various $\alpha \in (0, 1/2)$) for which $\log(A_{\lambda/\mu})$ is of order $\Theta(k^{3/2} n^{-1/2})$.
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$\rightarrow$ This bound is "sharp".

- When $k = \omega(n^{1/2})$, we proved $A_{\lambda/\mu} \leq \exp\left[k \log \frac{k^2}{n} + \mathcal{O}(k)\right]$.
Experimentally, $\log(A_{\lambda/\mu})$ is again at most of order $\Theta(k^{3/2} n^{-1/2})$.

Improving the bounds?

Conjecture (Dousse, F. ,'17)

There exists $C = C(L)$ such that for any balanced λ and μ, we have

$$\exp\left[-C\,k^{3/2}n^{-1/2}\right] \leq A_{\lambda/\mu} \leq \exp\left[C\,k^{3/2}n^{-1/2}\right],$$

- For $k = o(n^{1/3})$, this corresponds to our result;
- For $k = o(n^{1/2})$, we only have the upper bound;
- For $k = \omega(n^{1/2})$, we only have a weaker upper bound (and no lower bound).

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$$U_{\mathrm{best}}(\sigma,\nu) = \min\big(U_{\mathsf{R}}(\sigma,\nu), U_{\mathsf{LS}}(\sigma,\nu), U_{\mathsf{F\acute{S}}}(\sigma,\nu)\big),$$

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Proposition (Dousse, F.,'17)

$$\sum_{\sigma\in S_k} U_{best}(\sigma,\lambda)U_{best}(\sigma,\mu) \geq \exp\left[k \log \frac{k^2}{n} + \mathcal{O}(k)\right]$$

$\rightarrow$ even combining various bounds from the literature does not improve our result.

Symmetric shapes

Conjecture (Dousse, F.)

There exists $C = C(L)$ such that for any balanced λ and μ with either $\lambda' = \lambda$ or $\mu' = \mu$, we have

$$\exp\left[-Ck^2 n^{-1}\right] \leq A_{\lambda/\mu} \leq \exp\left[Ck^2 n^{-1}\right].$$

- Surprising from a combinatorial point of view; why does the symmetry of λ or μ change something?
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Thank you for your attention!