# Chaînes montantes-descendantes et limites d'échelle

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# Up-down chains

Let $\mathbb{S} = \bigcup_{n \geq 1} \mathbb{S}_n$ be a combinatorial class, with $|\mathbb{S}_1| = 1$. An updown chain is a Markov chain $p_n = p_n^{\uparrow} p_{n+1}^{\downarrow}$ on $\mathbb{S}_n$ consisting of

- an up-step $p_n^{\uparrow}$ from $\mathbb{S}_n$ to $\mathbb{S}_{n+1}$ (typically adding/duplicating an element);
- a down-step $p_{n+1}^{\downarrow}$ from $\mathbb{S}_{n+1}$ to $\mathbb{S}_{n+1}$ (typically deleting a random element).

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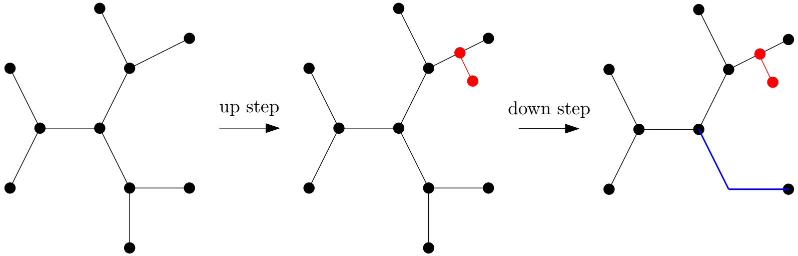
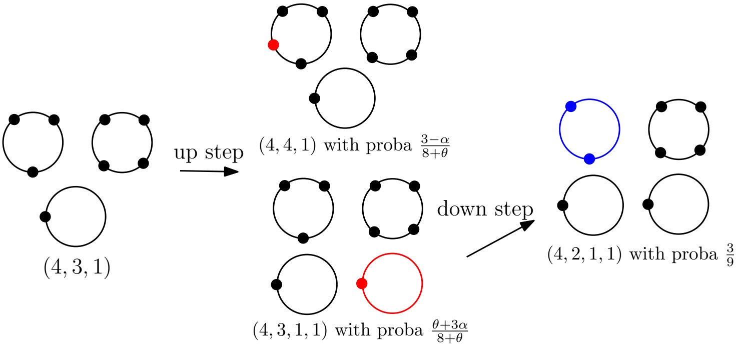
Motivations: tractable dynamic models, construction of diffusions on infinite-dimensional space states, Stein method.

# Example 1: trees (Aldous, 2000)




- up step: choose a uniform random edge, and attach to it a new leaf.
- down step: erase a uniform random leaf (and the corresponding edge and branching point).

# Example 2: partitions (Petrov 2009)




- up step: increase a part of size $i$ with probability $(i-\alpha)/(n+\theta)$, and create a new part with probability $(\theta+\alpha\ell)/(n+\theta)$, where $\ell$ is the number of parts.
  (For $\theta=1$, $\alpha=0$, this is a step of the Chinese Restaurant Process.)
- down step: remove a uniform random element (i.e. each part of size $i$ decreases with probability $i/(n+1)$).

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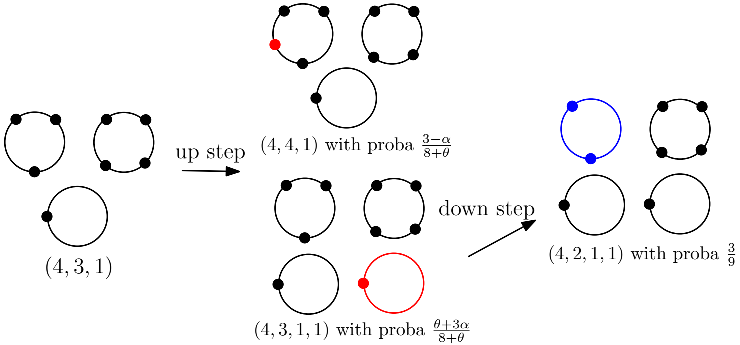
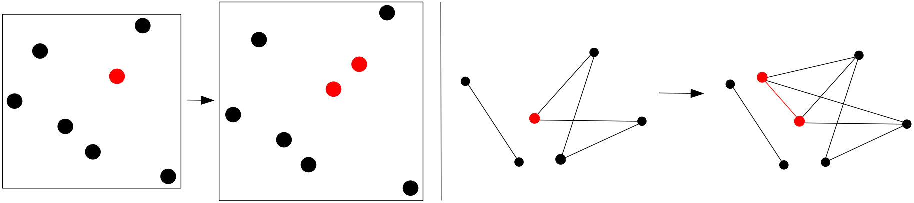



→ many variants in the literature:

- Involving Schur functions, $z$-measures on partitions/Thoma simplex (Borodin–Olshanski 2009), and Jack polynomials (Olshanski 2010);
- Strict partitions (Petrov 2010);
- Ordered version on integer compositions (Rivera-Lopez–Rizzolo 2022).

# Example 3: permutations/graphs

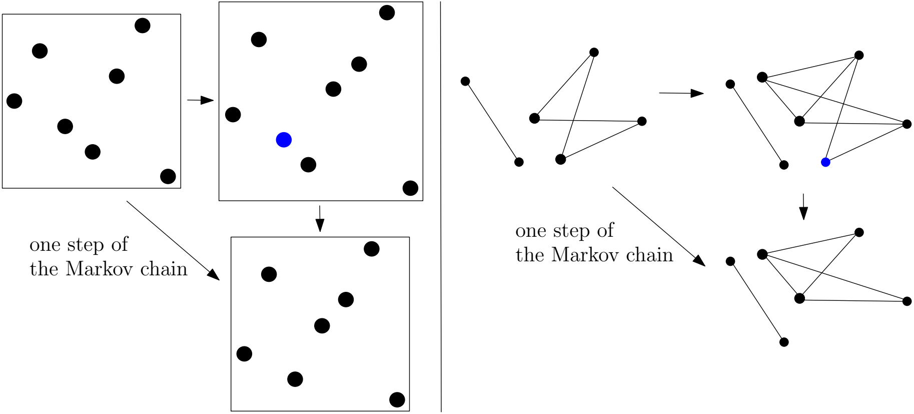


Upstep : duplicate a uniform random element/vertex.

With probability $p \in (0,1)$,

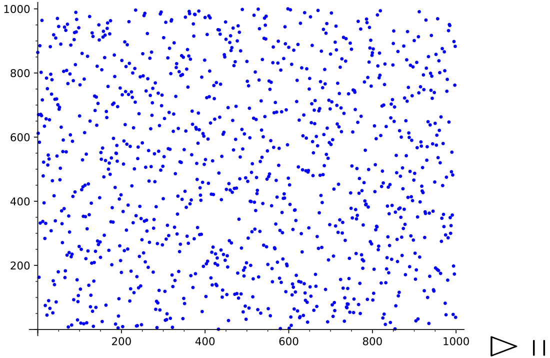
$$\begin{cases} \text{the "twin" elements are in increasing order (permutation case);} \\ \text{the two "twin" vertices are connected with probability } p \text{ (graph case).} \end{cases}.$$

# Example 3: permutations/graphs




Downstep: delete a uniform random element/vertex

# Example 3: permutations/graphs – simulation




Simulation of the up-down chain on permutations. Here, we take $q = 1/2$, $n = 1{,}000$, and we plot the permutation after $m$ steps, where $m \in \{0, \dots, 50\} \cdot 2{,}000$.

# Key assumption: the commutation relation

- Let $p_n^{\uparrow} \in \mathcal{M}(\mathbb{S}_n \times \mathbb{S}_{n+1})$ be the up transition matrix, i.e. $p_n^{\uparrow}(\tau,\sigma)$ is the probability to find $\sigma$ when duplicating a uniform random point in $\tau$.
- Let $p_{n+1}^{\downarrow} \in \mathcal{M}(\mathbb{S}_{n+1} \times \mathbb{S}_n)$ be the down transition matrix, i.e. $p_{n+1}^{\downarrow}(\sigma,\tau)$ is the probability to find $\tau$ when deleting a uniform random point in $\sigma$.

**Assumption (C)**

For any $n \geq 2$, we have

$$p_n^{\uparrow} p_{n+1}^{\downarrow} = \beta_n \, p_n^{\downarrow} p_{n-1}^{\uparrow} + (1-\beta_n)\, \mathrm{Id}_{\mathbb{S}_n},$$

Assumption (C) is fulfilled in the previous examples (with $\beta_n = \frac{n(2n-7)}{(n+1)(2n-5)}, \frac{n(n-1+\theta)}{(n+1)(n+\theta)}, \frac{n-1}{n+1}$ respectively).

(Intuition: adding and removing an element in different places commute, adding and removing an element in the same place gives $\mathrm{Id}_{\mathbb{S}_n}$.)

# Stationary distribution

## Proposition (general case)

*Assume (C). For* $s \in \mathbb{S}_n$*, let* $M_n(s) = p_1^{\uparrow} p_2^{\uparrow} \dots p_{n-1}^{\uparrow}(s_1, s)$*, where* $s_1$ *is the unique element of* $\mathbb{S}_1$*.*
*Then* $M_n$ *is the unique stationary measure of* $p_n$*.*

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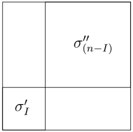
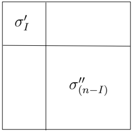
### Proposition (alternative description in the permutation case)

*For each* $k \geq 1$*, let* $\sigma_k, \sigma'_k, \sigma''_k$ *be independent random permutations with law* $M_k$*. Then, if* $I$ *is uniform in* $\{1, \dots, n-1\}$

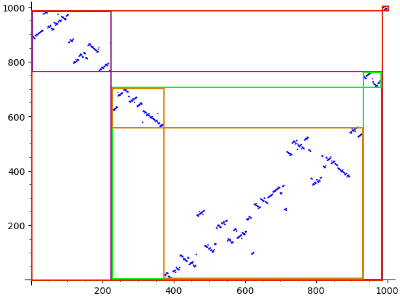
$$\mathrm{Law}(\sigma_n) = p\,\mathrm{Law}(\sigma'_I \oplus \sigma''_{(n-I)}) + (1-p)\,\mathrm{Law}(\sigma'_I \ominus \sigma''_{(n-I)}).$$

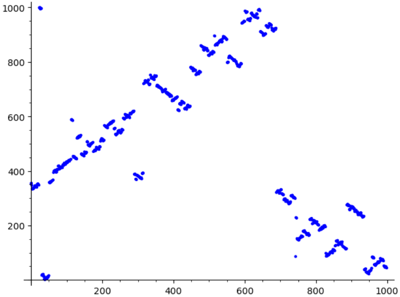
$\sigma'_I \oplus \sigma''_{(n-I)} =$ 


$\sigma'_I \ominus \sigma''_{(n-I)} =$ 


We call $\sigma_n$ the recursive separable permutation.

# Convergence to the stationary distribution - simulation




Left: Simulation of the stationary distribution ($n = 1000$), the colored square emphasizes the recursive structure of the limit.
Right: Simulation of the up-down chain on permutations after 250000 steps ($n = 1000$, $p = 1/2$).

# Separation distance (exact formula)

Definition (separation distance, Aldous–Diaconis, '87)

Let $(X(m))_{m\geq 0}$ be a Markov chain on a finite space $S$ with stationary distribution $M$

$$\Delta(m) := \max_{\substack{x,y\in S \\ M(y)\neq 0}} 1 - \frac{\mathbb{P}_x(X(m)=y)}{M(y)}.$$

It is a standard way to quantify speed of convergence for Markov chains.

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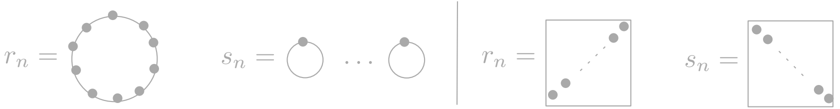
Assumption (S1): for each $n \geq 1$, there exist $r_n \neq s_n$ in $\mathbb{S}_n$ which are at distance $n-1$.

(Fulfilled in the partition and permutation examples



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**Proposition (F.–Rivera-Lopez, '25, based on Fulman, '09)**

*Assume (C) and (S1). Then, if $\Delta_n$ is the separation distance of $X_n$,*

$$\Delta_n(m) = \sum_{i=0}^{n-1} \left(1 - \frac{c_i}{c_n}\right)^m \prod_{\substack{0\leq j\leq n-1\\ j\neq i}} \frac{c_j}{c_j - c_i},$$

*where $c_n = (\beta_1 \dots \beta_n)^{-1}$, $c_n = \Theta(n^2)$ in the examples.*

# Density functions and eigenvalues of $p_n$

For $\tau$ in $\mathbb{S}_k$ and $\sigma$ in $\mathbb{S}_n$, with $k \le n$

$$d_\tau(\sigma) = (p_n^{\downarrow} \dots p_{k+1}^{\downarrow})(\sigma, \tau).$$

In words, $d_\tau(\sigma)$ is the probability to obtain $\tau$ when deleting $n-k$ uniform random elements in $\sigma$, or the "proportion of $\tau$" in $\sigma$.

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**Proposition (F., Rivera-Lopez, '25)**

*Under assumption (C), seeing $d_\tau$ as a vector in $\mathbb{C}^{\mathbb{S}_n}$,*

$$p_n \, d_\tau = (1 - \beta_k \cdots \beta_n) \, d_\tau + (\beta_k \cdots \beta_n) \sum_{\rho \nearrow \tau} p_{k-1}^{\uparrow}(\rho, \tau) d_\rho.$$

The eigenvalues of $p_n$ are $\lambda_k = 1 - \beta_k \cdots \beta_n$, with multiplicity $|\mathbb{S}_k| - |\mathbb{S}_{k-1}|$.
(Eigenvalues were known from Fulman, 2009, but without diagonal/triangular descriptions.)

# Scaling limit: assumption on limiting space

Informally, we assume that we have an inclusion $\mathbb{S} \hookrightarrow E$ in some space $E$, and that
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Examples :

- For permutations/graphs, such spaces are known and well-understood: permutons and graphons.
- In our partition example, $E$ is the Kingman simplex

$$\left\{(x_1 \geq x_2 \geq \dots), \sum x_i \leq 1\right\}.$$

- For trees, we need to use the space of algebraic trees introduced by Löhr–Mytnik–Winter, 2020 (it is a weaker topology than Gromov–Hausdorff convergence).

# Scaling limit result

Theorem (F., Rivera-Lopez, '25)

*Let $X_n$ be updown Markov chains satisfying assumption (C), and $E$ be an* appropriate *limiting space. Assume that $X_n(0)$ converge to $x$ in $E$.*

*Then there exists a Feller diffusion $F$ on $E$*

$$\big(X_n(c_n t\rfloor)\big)_{t\geq 0} \Longrightarrow \big(F(t)\big)_{t\geq 0},$$

*in distribution in the Skorokhod space $D([0,+\infty),\mathscr{P})$.*

*Moreover, the generator $\mathscr{A}$ of $F$ admits* $\mathrm{Span}(d_\tau, \tau\in\mathbb{S})$ *as a core, and we have, for $\tau$ in $\mathbb{S}_k$,*

$$\mathscr{A}\, d_\tau = -c_{k-1}\Big(d_\tau - \sum_{\rho\nearrow\tau} p^{\uparrow}_{k-1}(\rho,\tau)\, d_\rho\Big).$$

Generator of a process $F$: $\mathscr{A}\, g := \frac{d}{dt}\mathbb{E}[g(F(t))]\Big|_{t=0}$.

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→ unifies a number of previous results; new for the permutation/graph chain.

# Separation distance (asymptotics)

Theorem (F.–Rivera-Lopez, '25)

*Assume (C) and (S1), and in addition*

- $p_n^{\downarrow}(r_n, r_{n-1}) = 1$ *for* $n \geq 2$;
- $\sum_{n \geq 0} \frac{1}{c_n} < \infty$, *and that* $\{c_{n+1} - c_n\}_{n \geq 0}$ *is an unbounded, nondecreasing sequence.*

*Then*

$$\Delta_F(t) = \lim \Delta_n(\lfloor c_n t \rfloor) = \sum_{i=0}^{\infty} e^{-tc_i} \prod_{\substack{j=0 \\ j \neq i}}^{\infty} \frac{c_j}{c_j - c_i},$$

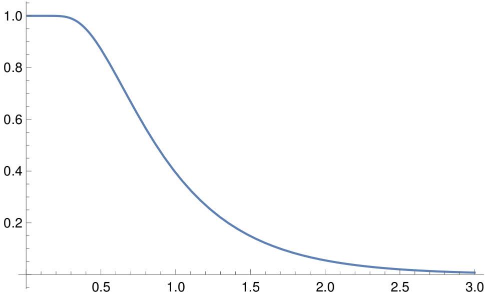
*where* $\Delta_F$ *is the separation distance of the limiting process* $F$.

# Asymptotics of the separation distance (permutation case)

**Example**

For the updown chain on permutations, we have

$$\lim_{n \to +\infty} \Delta_n(\lfloor n^2 t \rfloor) = \Delta_F(t) = \sum_{j=1}^{+\infty} (-1)^{j-1}(2j+1)e^{-tj(j+1)}.$$




Graph of $\Delta_F$.

# Thank you for your attention