Weighted dependency graphs

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Central limit theorems

Theorem

If $Y_1, Y_2, \dots$ are independent identically distributed variables with finite variance, and $X_n = \sum_{i=1}^n Y_i$, then

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\operatorname{Var} X_n}} \xrightarrow{d} \mathcal{N}(0,1). \tag{CLT}$$

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Goal of the talk: give an extension of dependency graphs that has a wide range of application.

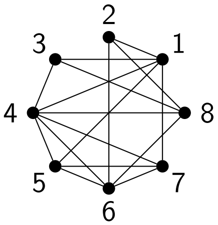
Dependency graphs

(Petrovskaya/Leontovich, Janson, Baldi/Rinott, Mikhailov, 80's)

A problem in random graphs

Erdős-Rényi model of random graphs $G(n, p)$:

- G has n vertices labelled $1, \ldots, n$;
- each edge $\{i, j\}$ is taken independently with probability p;



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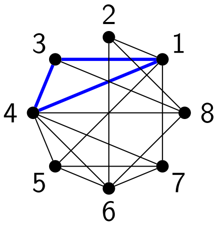

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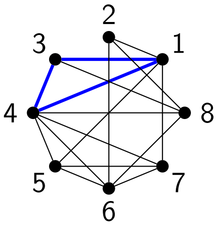

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$$T_n = \sum_{\Delta = \{i,j,k\} \subset [n]} Y_\Delta, \text{ where } Y_\Delta(G) = \begin{cases} 1 & \text{if } G \text{ contains the triangle } \Delta; \\ 0 & \text{otherwise.} \end{cases}$$

T_n is a sum of mostly independent variables.

Dependency graphs

Definition (Petrovskaya and Leontovich, 1982, Janson, 1988)

A graph L with vertex set A is a dependency graph for the family $\{Y_\alpha, \alpha \in A\}$ if

- if A_1 and A_2 are disconnected subsets in L, then $\{Y_\alpha, \alpha \in A_1\}$ and $\{Y_\alpha, \alpha \in A_2\}$ are independent.

Roughly: there is an edge between pairs of dependent random variables.

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Consider $G = G(n,p)$. Let $A = \{\Delta \in \binom{[n]}{3}\}$ (set of potential triangles) and

$$\{\Delta_1,\ \Delta_2\} \in E_L \text{ iff } \Delta_1 \text{ and } \Delta_2 \text{ share an edge in } G.$$

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Note: L has degree $O(n)$

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Janson's normality criterion

Setting: for each n,

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Theorem (Janson, 1988)

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For triangles, $N_n = \binom{n}{3}$, $\Delta_n = O(n)$, while $\sigma_n \asymp n^2$. (for fixed p)

Corollary

Fix p *in* $(0,1)$*. Then* T_n *satisfies a CLT.*

(also true for $p_n \to 0$ with $np_n \to \infty$; originally proved by Rucinski, 1988).

Applications of dependency graphs to CLT results

- mathematical modelization of cell populations (Petrovskaya, Leontovich, 82);
- subgraph counts in random graphs (Janson, Baldi, Rinott, Penrose, 88, 89, 95, 03);
- Geometric probability (Avram, Bertsimas, Penrose, Yukich, Bárány, Vu, 93, 05 , 07);
- pattern occurrences in random permutations (Bóna, Janson, Hitchenko, Nakamura, Zeilberger, 07, 09, 14).
- m-dependence (Hoeffding, Robbins, 53, ...; now widely used in statistics) is a special case.

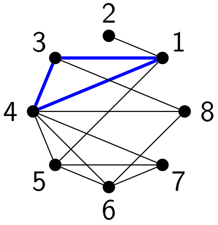
(Some of these applications use variants of Janson's normality criterion, which are more technical to state and omitted here...)

Not an application of dependency graphs

Random graph $G(n, M)$:

- G has n vertices labelled $1, \ldots, n$;
- The edge-set of G is taken uniformly among all possible edge-sets of cardinality M.

Example with $n = 8$ and $M = 14$



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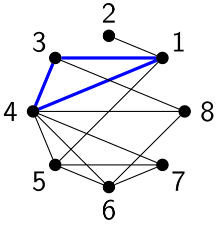

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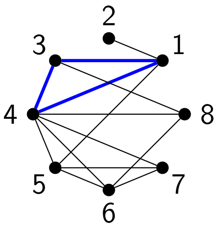

If $p = M / \binom{n}{2}$, each edge appears with probability p, but no independence any more!

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Fix $p \in (0; 1)$ and $M = p\binom{n}{2}$. Does the number of triangles T_n in $G(n, M_n)$ satisfy a CLT?

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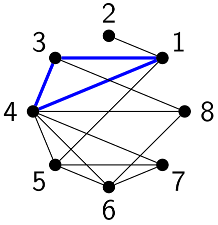

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Fix $p \in (0;1)$ and $M = p\binom{n}{2}$. Does the number of triangles T_n in $G(n, M_n)$ satisfy a CLT?

T_n still writes as a sum of Y_Δ, but the Y_Δ are pairwise dependent!

Solution: edge-weighted dependency graphs

weighted graphs = graphs with weights in $[0,1]$ on edges.

Definition

A weighted graph $\widetilde{L}$ with vertex set A is a weighted dependency graph for the family $\{Y_\alpha, \alpha \in A\}$ if

(*skipped for the moment*).

Roughly: the smaller the weight on the edge $\{Y_\alpha, Y_\beta\}$ is, the closer to independence Y_α and Y_β should be.

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$$\mathrm{wt}_{\widetilde{L}}(\{\Delta_1, \Delta_2\}) = \begin{cases} 1 & \text{if } \Delta_1 \text{ and } \Delta_2 \text{ share an edge in } G. \\ 1/n^2 & \text{otherwise.} \end{cases}$$

Then $\widetilde{L}$ is a weighted dependency graph for the family $\{Y_\Delta, \Delta \in \binom{[n]}{3}\}$.

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Note: $\widetilde{L}$ has degree $O(n^3)$, but **weighted** degree $O(n)$.

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A normality criterion for weighted dependency graphs

Setting: for each n,

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For triangles in $G(n, M_n)$, $N_n = \binom{n}{3}$, $\Delta_n = O(n)$, while $\sigma_n \asymp n^{3/2}$.

Corollary

Fix p *in* $(0,1)$ *and set* $M_n = p\binom{n}{2}$*. Then* T_n *satisfies a CLT.*

(also true for $n \ll M_n \ll n^2$; originally proved by Janson, 1994).

Applications of weighted dependency graphs

- crossings in random pair-partitions;
- subgraph counts in $G(n, M)$;
- random permutations;
- particles in symmetric simple exclusion process;
- subword counts in Markov chains;
- patterns in multiset permutations*, in set-partitions*;
- spins in Ising model*;
- determinantal point process**.

*in progress with Jehanne Dousse and Marko Thiel. **project
(Some of these applications use a variant of the above normality criterion, which is more technical to state...)

Cumulants

What are (mixed) cumulants?

- The r-th mixed cumulant κ_r of r random variables is a specific r-linear symmetric polynomial in joint moments. Examples:

$$\kappa_1(X) := \mathbb{E}(X), \quad \kappa_2(X,Y) := \mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

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- If a set of variables can be split in two mutually independent sets, then its mixed cumulant vanishes.

- If, for each r big enough, we have $\kappa_r(X_n) = o(\mathrm{Var}(X_n)^{r/2})$, then X_n satisfies a CLT. (Janson, 1988)

Sketch of proof of Janson's normality criterion

Setting: for each n,

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Fix $r \geq 1$. Then

$$\kappa_r(X_n) = \sum_{i_1,\dots,i_r} \kappa(Y_{n,i_1}, \cdots, Y_{n,i_r}).$$

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$$|\kappa_r(X_n)| \leq C_r N_n\, \Delta_n^{r-1}. \quad \square$$

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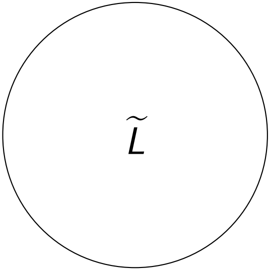

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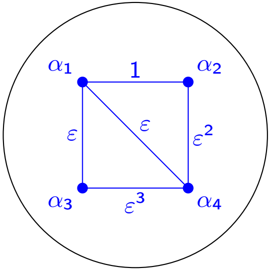

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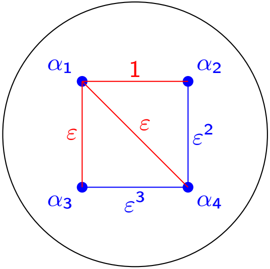
$\widetilde{L}[\alpha_1, \cdots, \alpha_r]$: graph induced by $\widetilde{L}$ on vertices $\alpha_1, \cdots, \alpha_r$.

$\mathcal{M}(K)$: Maximum weight of a spanning tree of K.

In the example, $\mathcal{M}\left(\widetilde{L}[\alpha_1, \cdots, \alpha_4]\right) = \varepsilon^2$.



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⚠ This is a simplified version of the definition; some of the applications need a more general but more technical version.

On the normality criterion for weighted dependency graphs

Proof is rather easy, similar as Janson's.
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How to prove that something is a weighted dependency graph for a family $\{Y_\alpha, \alpha \in A\}$?

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$\rightarrow$ in the next section, we give 3 general tools for that.

Finding weighted dependency graphs

Stability by powers

Setting:

- Let $\{Y_\alpha,\, \alpha \in A\}$ be r.v. with weighted dependency graph $\widetilde{L}$;
- fix an integer $m \geq 2$;
- for a multiset $B = \{\alpha_1, \cdots, \alpha_m\}$ of elements of A, denote

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Tool 1

The set of r.v. $\{\boldsymbol{Y}_B\}$ has a weighted dependency graph $\widetilde{L}^m$, where

$$\mathrm{wt}_{\widetilde{L}^m}(\boldsymbol{Y}_B, \boldsymbol{Y}_{B'}) = \max_{\alpha \in B, \alpha' \in B'} \mathrm{wt}_{\widetilde{L}}(Y_\alpha, Y_{\alpha'}).$$

In short: if we have a dependency graph for some variables Y_α, we have also one for monomials in the Y_α.

Back to triangles (1/2)

It is enough to find a weighted dependency graph for edge indicators:

$$Y_e = \mathbf{1}_{e \in G} \quad (e \in \binom{[n]}{2}).$$

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Tool 2 (informal version)

In (1), we can replace repeated variables or variables linked by edges of weight 1 by their products

$\rightarrow$ with Bernoulli variables, we can forget repetitions.

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We only need to prove

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Joint moments are explicit: let $E_n=\binom{n}{2}$,

$$J_\ell := \mathbb{E}(Y_{e_1}\dots Y_{e_\ell})=\frac{\binom{E_n-\ell}{M_n-\ell}}{\binom{E_n}{M_n}}=\frac{(E_n-\ell)!M_n!}{E_n!(M_n-\ell)!}.$$

Example: for $r=3$, we need to prove

$$J_3-3\,J_2\,J_1+2J_1^3=O(n^{-4}).$$

For fixed r, easy to check with a computer algebra system, but not easy to prove for general r.

The multiplicative criterion

Tool 3 (for edges in $G(n, M)$)

The bounds

$$\kappa\big(Y_{e_1}, \cdots, Y_{e_r}\big) = O(n^{-2(r-1)}) \quad \text{(for all } r \geq 1\text{)}$$

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Second statement is much easier to handle:

- it is "multiplicative" in J_i: can be done separately for each factorial factor.
- lots of cancellations in LHS.

$\rightarrow$ form here, quite easy to prove that we have a weighted dependency graph...

Other dependency graphs

Uniform random permutations

Let σ be a uniform random permutation of size n. Set

$$Y_{(i,s)} = \mathbf{1}_{\sigma(i)=s}.$$

Joint moment for distinct $i_1, \cdots, i_r$, $s_1, \cdots, s_r$:

$$\mathbb{E}\big(Y_{i_1,s_1} \cdots Y_{i_r,s_r}\big) = \frac{(n-r)!}{n!}.$$

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- It is a quotient of factorial factor, so it satisfies the multiplicity criterion.
- Thus we have a weighted dependency graphs for the $Y_{i,s}$;
- and, therefore, also for monomials in $Y_{i,s}$.

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$\rightarrow$ gives a bivariate extension of a functional CLT of Janson and Barbour.

Markov chains

Setting:

- Let $(M_i)_{i \geq 0}$ be an irreducible aperiodic Markov chain on a finite space state S;
- Assume M_0 is distributed with the stationary distribution π;
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Proposition

We have a weighted dependency graph $\widetilde{L}$ *with* $\mathrm{wt}_{\widetilde{L}}\left(\{Y_{i,s}, Y_{j,t}\}\right) = |\lambda_2|^{j-i}$, *where* λ_2 *is the second eigenvalue of the transition matrix.*

$\rightarrow$ CLT for linear statistics $\sum_{i=1}^{N} f(M_i) = \sum_{i,s} f(s) Y_{i,s}$.
Already known (huge literature on the subject).

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Corollary (using the stability by product)

We have a weighted dependency graph $\widetilde{L}^m$ for monomials $Y_{i_1,s_1}, \dots, Y_{i_m,s_m}$.

$\rightarrow$ gives a CLT for the number of copies of a given word in $(M_i)_{0 \leq i \leq N}$.
(Answers a question of Bourdon and Vallée.)

Discrete determinantal point processes

Setting: S discrete state space; X random subset of S.

Definition

X is a discrete determinantal point process (DPP) with kernel K if for any distinct $s_1, \dots, s_r$ in S,

$$\mathbb{P}(\{s_1, \dots, s_r\} \subseteq X) = \mathbb{E}\left(\prod_{i=1}^{r} \mathbf{1}_{s_i \in X}\right) = \det\Big(K(s_i, s_j)\Big)_{1 \le i,j \le r}.$$

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Strange definition (not even clear *a priori* if such a process exists at all), but there are lots of example:

- random Young diagrams, taken with Poissonized Plancherel measure;
- mid-time positions of non-intersecting random walks conditioned to come back to their starting positions.
- eigenvalues of random matrices (continuous DPP);

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Lemma (Soshnikov, 2000)

If X is a discrete determinantal point process with kernel K, then, for any distinct $s_1, \dots, s_r$ in S,

$$\kappa(\mathbf{1}_{s_1 \in X}, \dots, \mathbf{1}_{s_r \in X}) = \sum_\sigma \varepsilon(\sigma) \prod_i K(s_i, s_{\sigma(i)}),$$

where the sum runs over cyclic permutation in S_r.

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Soshnikov cumulant formula $\Rightarrow$ for each DPP, we have a weighted dependency graph with weights $K(s_i, s_j)$.

again, CLT for linear statistic is known;
Project: investigate CLT for "multilinear" statistics.

Conclusion

- We provide a general tool to prove CLT for sums of weakly dependent random variables.
- Other examples (d-regular graphs)?
- Can we prove other type of results: speed of convergence, large deviations, ...? (with P.-L. Méliot and A. Nikeghbali, we have such results for usual dependency graphs.)
- A theory of continuous weighted dependency graphs (to handle continuous determinantal point processes)?