Cumulants of Jack symmetric functions and b-conjecture

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Part 1: Background and main result

Jack polynomials

- Family $J_\lambda^{(\alpha)}(\boldsymbol{x})$ of symmetric functions depending on a **parameter $\alpha > 0$**;
- deformation of **Schur** and **elementary** symmetric functions:

$$J_\lambda^{(1)}(\boldsymbol{x}) = \frac{n!}{\dim(\lambda)} s_\lambda(\boldsymbol{x}),$$

$$J_\lambda^{(0)}(\boldsymbol{x}) = \left(\prod_i \lambda_i^t!\right) e_{\lambda^t}(\boldsymbol{x});$$

- For $\alpha = 2$, we get **zonal** symmetric functions, that have a representation-theoretical interpretation.
- special case of **Macdonald** polynomials:

$$\lim_{t\to 1} (1-t)^{-|\lambda|} J_\lambda^{(t^\alpha,t)}(\boldsymbol{x}) = J_\lambda^{(\alpha)}(\boldsymbol{x});$$

- widely studied since their introduction by Jack in 1970.

b-conjecture I

Goulden and Jackson [3] defined a family of coefficients $\boldsymbol{h^\tau_{\mu,\nu}(\alpha-1)}$ by the following identity:

$$\log\left(\sum_{n\geq 1}\sum_{\lambda\vdash n} \frac{J_\lambda^{(\alpha)}(\boldsymbol{x})\, J_\lambda^{(\alpha)}(\boldsymbol{y})\, J_\lambda^{(\alpha)}(\boldsymbol{z})\, t^n}{h_\alpha(\lambda)\, h'_\alpha(\lambda)}\right) =$$

$$\sum_{n\geq 1} \frac{t^n}{\alpha\, n}\left(\sum_{\mu,\nu,\tau\vdash n} \boldsymbol{h^\tau_{\mu,\nu}(\alpha-1)}\, p_\mu(\boldsymbol{x})\, p_\nu(\boldsymbol{y})\, p_\tau(\boldsymbol{z})\right),$$

where $h_\alpha(\lambda)$ and $h'_\alpha(\lambda)$ are α-deformation of the "hook-product". Motivation:

- $h^\tau_{\mu,\nu}(0)$ enumerates connected hypergraphs embedded into **oriented surfaces** with particular statistics given by μ, ν and τ,
- $h^\tau_{\mu,\nu}(1)$ enumerates connected hypergraphs embedded into **non-oriented surfaces** with the same statistics.

b-Conjecture II

b-Conjecture (Goulden, Jackson)

For all partitions $\tau, \mu, \nu \vdash n \geq 1$, the quantity $h^\tau_{\mu,\nu}(\beta)$ is a **polynomial** in β with **nonnegative integer** coefficients.

A previous result of us states that $h^\tau_{\mu,\nu}(\beta)$ is a rational function in α with only possible poles at $\alpha = 0$. Our main result is a proof that, in fact, $h^\tau_{\mu,\nu}(\beta)$ has no pole at $\alpha = 0$, thus completing the proof of **polynomiality** in b-Conjecture:

Theorem (Main Result)

For all partitions $\tau, \mu, \nu \vdash n \geq 1$ quantity $h^\tau_{\mu,\nu}(\beta)$ is a **polynomial** in β of degree $2 + n - \ell(\tau) - \ell(\mu) - \ell(\nu)$ with rational coefficients.

Part 2: Strong factorization property

Strong factorization property of Jack polynomials

Our main result is a consequence of the **strong factorization property (SFP)** of Jack polynomials:

- for partitions $\lambda = (\lambda_1, \lambda_2, \dots)$, and $\mu = (\mu_1, \mu_2, \dots)$ we define

$$\lambda \oplus \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots);$$

- for partitions $\lambda^1, \cdots, \lambda^r$ and a subset I of $[r] := \{1, \cdots, r\}$, we denote

$$\lambda^I := \bigoplus_{i\in I} \lambda^i.$$

Theorem (SFP for Jack)

Let $\lambda^1, \cdots, \lambda^r$ be partitions. Then

$$\prod_{I\subseteq[r]} \left(J^{(\alpha)}_{\lambda^I}\right)^{(-1)^{|I|}} = 1 + O(\alpha^{r-1}).$$

Examples:

$$\frac{J^{(\alpha)}_{\lambda^1\oplus\lambda^2}}{J^{(\alpha)}_{\lambda^1} J^{(\alpha)}_{\lambda^2}} = 1 + O(\alpha), \qquad \frac{J^{(\alpha)}_{\lambda^1\oplus\lambda^2\oplus\lambda^3} J^{(\alpha)}_{\lambda^1} J^{(\alpha)}_{\lambda^2} J^{(\alpha)}_{\lambda^3}}{J^{(\alpha)}_{\lambda^2\oplus\lambda^3} J^{(\alpha)}_{\lambda^1\oplus\lambda^3} J^{(\alpha)}_{\lambda^1\oplus\lambda^2}} = 1 + O(\alpha^2).$$

The strong factorization property can also be stated in terms of **cumulants**:

Theorem (SFP for Jack, 2nd form)

For any partitions $\lambda^1, \dots, \lambda^r$, one has

$$\sum_{\pi\in\mathcal{P}([r])} (-1)^{|\pi|}(|\pi|-1)! \prod_{B\in\pi} J_{\lambda^B} = O(\alpha^{r-1}).$$

Examples:

- $J^{(\alpha)}_{\lambda^1\oplus\lambda^2} - J^{(\alpha)}_{\lambda^1} J^{(\alpha)}_{\lambda^2} = O(\alpha)$,
- $J^{(\alpha)}_{\lambda^1\oplus\lambda^2\oplus\lambda^3} - J^{(\alpha)}_{\lambda^1} J^{(\alpha)}_{\lambda^2\oplus\lambda^3} - J^{(\alpha)}_{\lambda^2} J^{(\alpha)}_{\lambda^1\oplus\lambda^3}$ $- J^{(\alpha)}_{\lambda^3} J^{(\alpha)}_{\lambda^1\oplus\lambda^2} + 2 J^{(\alpha)}_{\lambda^1} J^{(\alpha)}_{\lambda^2} J^{(\alpha)}_{\lambda^3} = O(\alpha^2)$.

Notation: if F is a function on Young diagrams, $\kappa^F(\lambda^1, \cdots, \lambda^r)$ denotes $\sum_{\pi\in\mathcal{P}([r])}(-1)^{|\pi|}(|\pi|-1)!\prod_{B\in\pi} F(\lambda^B)$.

Two equivalent SFP

Let R- ring, and $\boldsymbol{u} = (u_I)_{I\subseteq[r]}$ - family of elements of $R(\alpha)$ indexed by subsets of $[r]$.

Proposition (F, 2013)

Assume $u_{\{h\}}, u^{-1}_{\{h\}} = O(1)$. The following are equivalent:

- for all $H \subseteq [r]$, $\prod_{G\subseteq H} u_G^{(-1)^{|H|}} = 1 + O(\alpha^{|H|-1})$;
- for all $H \subseteq [r]$, $\sum_{\pi\in\mathcal{P}(H)}(-1)^{|\pi|}(|\pi|-1)!\prod_{B\in\pi} u_B = O(\alpha^{|H|-1})$.

If this holds, we say that $\boldsymbol{u}$ has the **strong factorization property (SFP)**.

Corollary:

For two families $(u_I)_{I\subseteq[r]}$ and $(v_I)_{I\subseteq[r]}$ with SFP, their entry-wise product $(u_I v_I)_{I\subseteq[r]}$ and quotient $(u_I/v_I)_{I\subseteq[r]}$ also have SFP.

Back to our main theorem

Lemma

$$\log\left(\sum_{n\geq 1}\sum_{\lambda\vdash n} \frac{J_\lambda^{(\alpha)}(\boldsymbol{x})\, J_\lambda^{(\alpha)}(\boldsymbol{y})\, J_\lambda^{(\alpha)}(\boldsymbol{z})\, t^n}{h_\alpha(\lambda)\, h'_\alpha(\lambda)}\right) =$$

$$\sum_{r\geq 1} \frac{t^r}{r!\alpha^r} \sum_{(j_1,\cdots,j_r)} \kappa^G(1^{j_1}, \cdots, 1^{j_r}),$$

where $G(\lambda) = \frac{1}{h_\alpha(\lambda) h''_\alpha(\lambda)} J^\alpha_\lambda(\boldsymbol{x}) J^\alpha_\lambda(\boldsymbol{y}) J^\alpha_\lambda(\boldsymbol{z})$.

Thus our main theorem is equivalent to:

$$\kappa^G(1^{j_1}, \cdots, 1^{j_r}) = O(\alpha^{r-1}).$$

For any $r \geq 1$ and for any partitions $\lambda^1, \cdots, \lambda^r$,

- the family $u_I := J^\alpha_{\lambda^I}(\boldsymbol{x})$ has SFP,
- the family $v_I := h_\alpha(\lambda^I)$ has SFP,
- the following family also has SFP: $w_I := h''_\alpha(\lambda^I) := \alpha^{-\lambda_1}\left(\prod_i m_i(\lambda^t)!\right)^{-1} h'_\alpha(\lambda)$.
- as a consequence, the family $u_I/(v_I w_I) = G(\lambda^I)$ has SFP. $\square$

Conclusion

- Strong factorization property of Jack polynomials leads to **polynomiality result in b-conjecture**;
- Polynomiality result in b-conjecture leads to combinatorial interpretation of the top-degree part of the coefficients $h^\tau_{\mu,\nu}(\beta)$ with $\tau = (n)$, and to the proof of b-conjecture in some special cases [1].

Open questions

Conjecture (SFP for Macdonald)

Let $\lambda^1, \cdots, \lambda^r$ be partitions, and $J^{(q,t)}_\lambda$ be Macdonald polynomials. Then

$$\prod_{I\subseteq[r]} \left(J^{(q,t)}_{\lambda^I}\right)^{(-1)^{|I|}} = 1 + O\big((q-1)^{r-1}\big).$$

Equivalently,

$$\sum_{\pi\in\mathcal{P}([r])} \mu(\pi, \{H\}) \prod_{B\in\pi} J^{(q,t)}_{\lambda^B} = O\big((q-1)^{r-1}\big).$$

Question

Can we find a probabilistic framework in which the strong factorization property of Jack polynomials leads to some kind of central limit theorem?

Thank you

Thank you for your attention. Here is a list of references for more on the subject.

References

[1] M. Dołęga. Top degree part in b-conjecture for unicellular bipartite maps. arXiv preprint 1604.03288, 2016.

[2] M. Dołęga and V. Féray. Cumulants of Jack symmetric functions and b-conjecture. arXiv preprint 1601.01501, 2016.

[3] I. P. Goulden and D. M. Jackson. Connection coefficients, matchings, maps and combinatorial conjectures for Jack symmetric functions. *Trans. Amer. Math. Soc.*, 348(3):873–892, 1996.