What do random constraint permutations look like?

Valentin Féray
(joint work with F. Bassino, M. Bouvel,
L. Gerin, M. Maazoun and A. Pierrot)

Institut für Mathematik, Universität Zürich

Colloquium, University of Fribourg,
Dec. 5th, 2017

Introduction

Main topic: random permutations

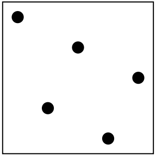
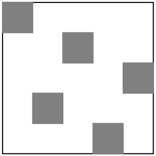
- Classical questions: look at some statistics, like the number of cycles (of given length), longest increasing subsequences, ...
  (usually for uniform or Ewens distributions)

- a more recent approach: look for a limit theorem for the permutation itself (interesting for non-uniform models or constrained permutations).

The theory of permutons
(Hoppen, Kohayakawa, Moreira, Rath, Sampaio)

How to look at large permutations?

A permutation π can be encoded as a probability measure μ_π on $[0,1]^2$.

$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

In μ_π, each small square has weight $1/n$ (i.e. density n).

We have a natural notion of limit for such objects: the weak convergence.
This defines a nice compact Polish space.

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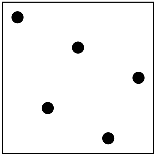
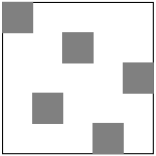
$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

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Note: the projection on μ_π on each axis is the Lebesgue measure on $[0,1]$ (in other words, μ_π has uniform marginals).
$\rightarrow$ potential limits also have uniform marginals.

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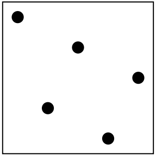
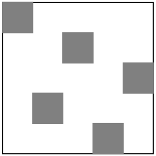
$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

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Definition

A permuton is a probability measure on $[0,1]^2$ with uniform marginals.

Next few slides: connection with permutation patterns.

Permutation patterns

Definition

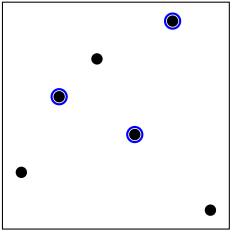
An occurrence of a pattern τ in σ is a subsequence $\sigma_{i_1} \dots \sigma_{i_k}$ that is order-isomorphic to τ, *i.e.* $\sigma_{i_s} < \sigma_{i_t} \;\Leftrightarrow\; \tau_s < \tau_t$.

Example (occurrences of 2 1 3)

2 4 5 3 6 1

8 2 3 4 6 1 7 5

Visual interpretation



Pattern density in permutations and permutons

If τ and σ are permutations of size k and n, resp., we set

$$\widetilde{\mathrm{occ}}(\tau,\sigma) := \binom{n}{k}^{-1} \cdot \# \left\{ \begin{array}{c} \text{occurrences of} \\ \tau \text{ in } \sigma \end{array} \right\} \in [0,1].$$

In other terms: take k elements uniformly at random in σ, the probability to find a pattern τ is $\widetilde{\mathrm{occ}}(\tau,\sigma)$.

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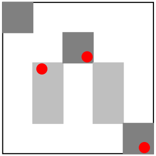
This probabilistic interpretation extends to permutons: replacing σ with a permuton μ

$$\widetilde{\mathrm{occ}}(\tau,\mu) := \mathbb{P}^{\mu}(U^{(1)},\cdots,U^{(k)} \text{ form a pattern } \tau),$$

where $U^{(1)},\cdots,U^{(k)}$ are i.i.d. points in $[0,1]^2$ with distribution μ.



a "231 pattern" in a permuton

An approximation lemma

Reminder:

$$\widetilde{\mathrm{occ}}(\tau,\sigma) := \binom{n}{k}^{-1} \cdot \# \left\{ \begin{array}{c} \text{occurrences of} \\ \tau \text{ in } \sigma \end{array} \right\} \in [0,1].$$

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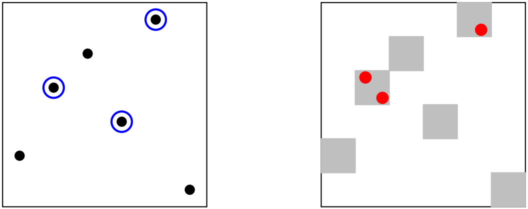
⚠ In general, $\widetilde{\mathrm{occ}}(\tau,\sigma) \neq \widetilde{\mathrm{occ}}(\tau,\mu_\sigma)$.



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But we have the following approximation lemma:

Lemma

If π and σ are permutations of size k and n, resp., then

$$|\,\widetilde{\mathrm{occ}}(\pi,\sigma) - \widetilde{\mathrm{occ}}(\pi,\mu_\sigma)| \leq \frac{1}{n}\binom{k}{2}.$$

Pattern density convergence and permuton convergence

Theorem (Hoppen, Kohayakawa, Moreira, Rath, Sampaio, 2013)

Weak convergence of permutons is equivalent to the pointwise convergence of $\widetilde{\mathrm{occ}}(\tau, \cdot)$ *for all* τ, i.e.

$$\mu^{(n)} \to \mu \;\Leftrightarrow\; \textit{for all } \tau,\; \widetilde{\mathrm{occ}}(\tau, \mu^{(n)}) \to \widetilde{\mathrm{occ}}(\tau, \mu).$$

As a consequence, for a sequence of permutation $\sigma^{(n)}$ of size tending to infinity,

$$\mu_{\sigma^{(n)}} \to \mu \;\Leftrightarrow\; \text{for all } \tau,\; \widetilde{\mathrm{occ}}(\tau, \sigma^{(n)}) \to \widetilde{\mathrm{occ}}(\tau, \mu).$$

(In terms of permutations, $\widetilde{\mathrm{occ}}(\tau, \sigma^{(n)})$ is much more concrete!)

Proof: $\Rightarrow$ is easy, $\Leftarrow$ see next two slides.

Proof that $\forall \tau, \widetilde{occ}(\tau, \mu^{(n)}) \to \widetilde{occ}(\tau, \mu) \;\Rightarrow\; \mu^{(n)} \to \mu$ (1/2)

Consider three independent r.v. (U_1, U_2), (V_1, V_2) and (W_1, W_2) in $[0,1]^2$ according to μ and let

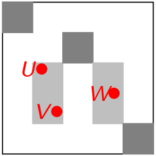
$$p_\mu := \mathbb{P}(U_1 < W_1, V_2 < W_2).$$

- Integrating over the value of (W_1, W_2), we get

$$p_\mu = \int_{[0,1]^2} \mathbb{P}(U_1 < x,\ V_2 < y)\, d\mu(x,y) = \int_{[0,1]^2} x\, y\, d\mu(x,y).$$

- On the other hand, we can split the event $\{U_1 < W_1, V_2 < W_2\}$ depending on whether $U_1 < V_1$ or $U_1 > V_1$, whether $V_1 < W_1$ or $V_1 > W_1$, ...

$$p_\mu = \mathbb{P}(U_1 < V_1 < W_1, V_2 < W_2 < U_2)$$
$$+ \text{ other terms of the same kind;}$$
$$= \tfrac{1}{6}\, \widetilde{occ}(312, \mu) + \dots.$$



Proof that $\forall \tau, \widetilde{\mathrm{occ}}(\tau, \mu^{(n)}) \to \widetilde{\mathrm{occ}}(\tau, \mu) \; \Rightarrow \; \mu^{(n)} \to \mu$ (2/2)

Conclusion: the map $\mu \mapsto p_\mu = \int_{[0,1]^2} x\, y\, d\mu(x,y)$ is a linear combination of maps $\mu \mapsto \widetilde{\mathrm{occ}}(\tau, \mu)$ for τ in S_3.

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Bold generalization: for any p, q there exists constants $c^{\tau}_{p,q}$ such that for all permutons μ,

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Then convergence of all $\widetilde{\mathrm{occ}}(\tau, \cdot)$ implies moment convergence, which in turn implies convergence in distribution.

Permuton convergence of random permutations

Theorem (BBFGMP, 17)

Let $\boldsymbol{\sigma}_n$ be a random permutation of size n. The following assertions are equivalent.

(a) $\mu_{\boldsymbol{\sigma}_n}$ *converges in distribution for the weak topology to some random permuton* $\boldsymbol{\mu}$.

(b) *The random infinite vector* $\left(\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)\right)_{\pi \in \mathfrak{S}}$ *converges in distribution in the product topology to some random infinite vector* $(\boldsymbol{\Lambda}_\pi)_{\pi \in \mathfrak{S}}$.

(c) *For every* π *in* $\mathfrak{S}$, *there is a* $\Delta_\pi \geq 0$ *such that*

$$\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)] \xrightarrow{n\to\infty} \Delta_\pi.$$

Note: $(a) \Leftrightarrow (b)$ expected (random version of the previous result),
$(b) \Leftrightarrow (c)$ might be more surprising (cv in expectation is enough!).

Why are expectations enough?

Claim: Fix $\tau_1, \dots, \tau_k$. There exist constants c_ρ such that, for all permutons μ,

$$\prod_{i=1}^{k} \widetilde{\text{occ}}(\tau_i, \mu) = \sum_{\rho} c_\rho \, \widetilde{\text{occ}}(\rho, \mu).$$

(Similar argument as in the previous proof.)

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(Similar argument as in the previous proof.)

Hence the cv in expectation implies cv of joint moments, which is enough to deduce cv in distribution (our random variables are bounded).

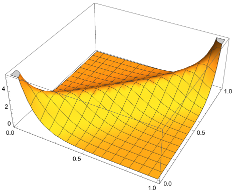
First examples of permuton convergence

Mallows model on S_n: $\mathbb{P}(\sigma_n) \propto q_n^{\mathrm{inv}(\sigma_n)}$,
where $\mathrm{inv}(\sigma) = \#\{(i,j) \text{ with } i < j \text{ and } \sigma(i) > \sigma(j)\}$.

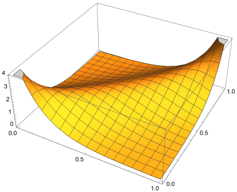
Theorem (Starr, '09)

Take $q_n = 1 - \beta/n$. Then $\mu_{\boldsymbol{\sigma}^{(n)}}$ converge to the deterministic permuton with density

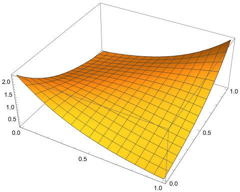
$$u(x,y) = \frac{(\beta/2)\sinh(\beta/2)}{\left(e^{\beta/4}\cosh(\beta[x-y]/2) - e^{-\beta/4}\cosh(\beta[x+y-1]/2)\right)^2}.$$



$\beta = 10$



$\beta = 6$



$\beta = 2$

Random minimal factorizations (Angel, Holroyd, Romik, Virag, '06)

Consider a uniform random minimal factorization of $\omega_0 := n$ n-1 ... 2 1 into adjacent transpositions: $\omega_0 = \tau_1 \dots \tau_N$ (where $N = \binom{n}{2}$).
Q: what do partial products $\tau_1 \dots \tau_{\lfloor cN \rfloor}$ look like?

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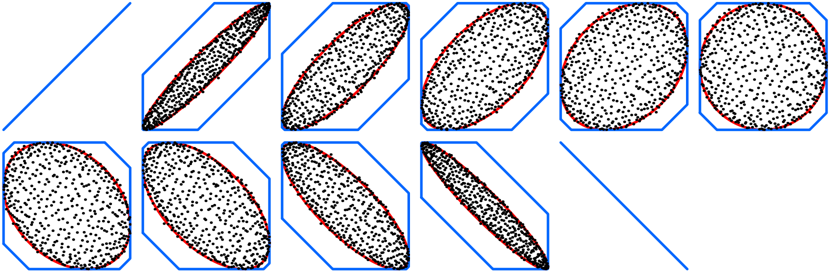
Pictures (©AHRV) ($n = 500,\ c = 0, .1, .2, \dots, .9, 1$):



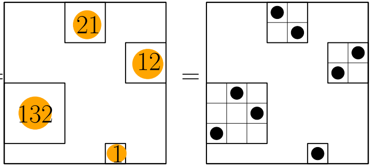
There is a conjectural formula for the limiting process in the space of permutons.

Substitution-closed classes and Brownian permutons

Substitution in permutations

Definition

Let θ be a permutation of size d and $\pi^{(1)}, \dots, \pi^{(d)}$ be permutations. The diagram of the permutation $\theta[\pi^{(1)}, \dots, \pi^{(d)}]$ is obtained by replacing the i-th dot in the diagram of θ with the diagram of $\pi^{(i)}$ (for each i).

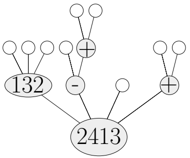
$2413[132, 21, 1, 12] =$  $= 24387156$

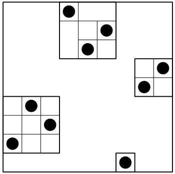
$\rightarrow$ we are interested in substitution-closed permutation classes $\mathcal{C}$, i.e. set $\mathcal{C} \subsetneq \biguplus_{n \geq 0} S_n$ such that:

- $\theta, \pi^{(1)}, \dots, \pi^{(d)} \in \mathcal{C} \Rightarrow \theta[\pi^{(1)}, \dots, \pi^{(d)}] \in \mathcal{C}$;
- τ occurs in π and $\pi \in \mathcal{C} \Rightarrow \tau \in \mathcal{C}$.

Tree representation in substitution closed classes (Albert, Atkinson, '05)

Permutations in a substitution closed class $\mathcal{C}$ can be represented by "substitution trees":





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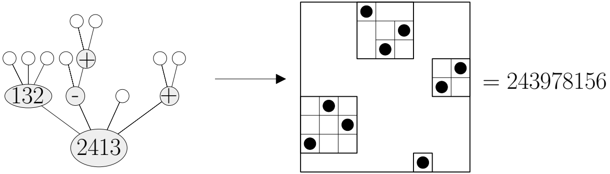

A priori this tree representation is not unique, but it can be made unique by imposing constraints (in particular, node labels should be $\oplus$, $\ominus$ or *simple permutations*).

$\rightarrow$ the set $\mathcal{S}$ of simple permutations in $\mathcal{C}$ will play a crucial role.

Limit of substitution-closed classes

Theorem (BBFGMP, '17)

Let $\mathcal{C}$ be a substitution-closed class whose set of simple permutations $\mathcal{S}$ has generating function $S(z) = \sum_{\alpha \in \mathcal{S}} z^{|\alpha|}$. Assume

$$R_S > 0 \quad \textit{and} \quad S'(R_S) > \frac{2}{(1+R_S)^2} - 1. \tag{H1}$$

R_S: radius of convergence of $S(z)$.

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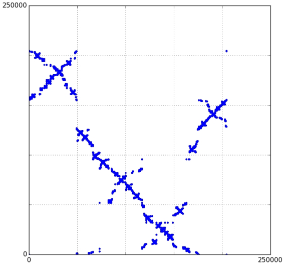
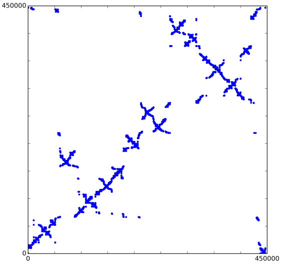
For every $n \geq 1$, let $\boldsymbol{\sigma}_n$ be a uniform permutation in $\mathcal{C}$. The sequence $(\mu_{\boldsymbol{\sigma}_n})_n$ tends to the biased Brownian separable permuton $\boldsymbol{\mu}^{(p)}$ for some "explicit" parameter p in $[0,1]$.

- $\boldsymbol{\mu}^{(p)}$ (and $\widetilde{\mathrm{occ}}(\tau, \mu^{(p)})$) can be constructed from the Brownian excursion or the continuous Brownian tree $\mathcal{T}$.
- First example of non-deterministic permuton limits;
- universality phenomenon: the limit only depends on $\mathcal{S}$ through a single parameter p (in practice, always closed to $1/2$).

Pictures



Simulation of $\boldsymbol{\sigma}_n$ with
$\mathcal{S} = \emptyset$
(separable permutations)



Simulation of $\boldsymbol{\sigma}_n$ with
$\mathcal{S} = \{2413, 3142, 24153\}$

Other limiting behaviours

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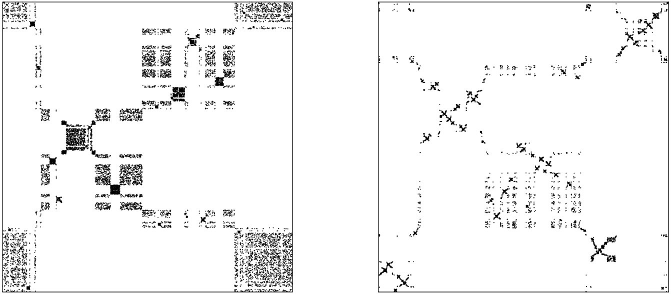
Intuition: in case 2, the tree encoding $\boldsymbol{\sigma}_n$ has one vertex of very large degree. In case 3b., it tends towards a stable tree.

(Cases 2, 3a and 3b require additional technical hypotheses.)

Pictures (stable permutons)



Simulation of stable permutons of parameter $\delta = 1.1$ and $\delta = 1.5$

☹ We do not know substitution-closed classes which fits in case 3b. This simulation is a ad-hoc model constructed to converge towards the stable permutons.

A word on the proofs

1. Reminder: enough to prove that, for any τ,
$$\mathbb{E}\big[\,\widetilde{\mathrm{occ}}(\tau, \boldsymbol{\sigma}_n)\big] \to \mathbb{E}\big[\,\widetilde{\mathrm{occ}}(\tau, \boldsymbol{\nu})\big],$$
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2. The LHS can be computed combinatorially:
$$\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)] = \frac{\#\{\sigma \in \mathcal{C}_n, I \subset [n] : \mathrm{pat}_I(\sigma) = \pi\}}{\binom{n}{k}\ \mathcal{C}_n}.$$
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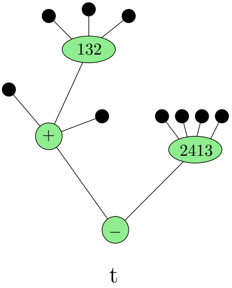
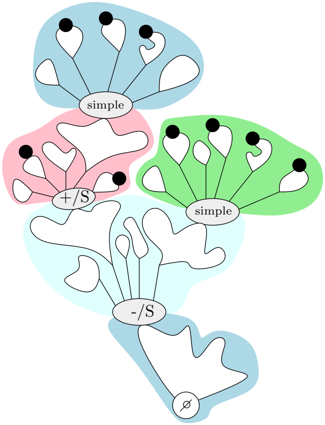
3. The RHS can be evaluated through the theory of random trees.

Combinatorial decomposition of canonical trees with marked leaves inducing a given t





The white pieces are trees with zero or one marked leaf and some conditions (to avoid creating adjacent $\oplus$ by gluing).

Translating that into equations (1/2)

Equations for the white pieces:

- One implicit equation

$$T_{\mathrm{not}\oplus} = z + \frac{T_{\mathrm{not}\oplus}^2}{1 - T_{\mathrm{not}\oplus}} + S\left(\frac{T_{\mathrm{not}\oplus}}{1 - T_{\mathrm{not}\oplus}}\right).$$

- Other series are expressed in terms of this one

$$\begin{aligned} T &= \frac{T_{\mathrm{not}\oplus}}{1 - T_{\mathrm{not}\oplus}}; \\ T^+ &= \frac{1}{1 - WS'(T) - W - S'(T)}; \\ T^+_{\mathrm{not}\ominus} &= \frac{1}{1+W}\, T^+; \\ T^+_{\mathrm{not}\oplus} &= (WS'(T) + W + S'(T))\, T^+_{\mathrm{not}\ominus} \end{aligned}$$

where $W = (\frac{1}{1 - T_{\mathrm{not}\oplus}})^2 - 1$.

Translating that into equations (2/2)

Equation for $\mathrm{Num}^{(t)}(z) = \sum \mathrm{Num}_n^{(t)} z^n$:

$$\mathrm{Num}^{(t)}(z) = z^k \sum_{V_s} T^{\text{type of root}} \prod_{v \in \mathrm{Int}(t)} A_v,$$

where

$$A_v = \begin{cases} \mathrm{Occ}_{\theta_v}(T)\ (T')^{d'_v} (T^+)^{d^+_v} (T^-)^{d^-_v} & \text{if } v \in V_s\,, \\ \left(\frac{1}{1-T_{\mathrm{not}\oplus}}\right)^{d_v+1} (T'_{\mathrm{not}\oplus})^{d'_v} (T^+_{\mathrm{not}\oplus})^{d^+_v} (T^-_{\mathrm{not}\oplus})^{d^-_v} & \text{if } v \notin V_s \text{ and } \theta_v = \oplus\,, \\ \left(\frac{1}{1-T_{\mathrm{not}\ominus}}\right)^{d_v+1} (T'_{\mathrm{not}\ominus})^{d'_v} (T^+_{\mathrm{not}\ominus})^{d^+_v} (T^-_{\mathrm{not}\ominus})^{d^-_v} & \text{if } v \notin V_s \text{ and } \theta_v = \ominus\,. \end{cases}$$

Second step: singularity analysis

1. Find singularity exponents: the singular part of all these series is $\mathrm{cst}(1 - \frac{z}{\rho})^{\beta}(1 + o(1))$ where β is:

	Brownian case	stable case	degenerate case
simple permutations	analytic	$\delta \in (1, 2)$	$\delta > 1$
canonical trees	$1/2^*$	$1/\delta$	δ
trees with one marked leaf	$-1/2$	$\frac{1}{\delta} - 1$	$\delta - 1$
$\mathrm{Num}^{(t)}(z)$	$-(e + 1)/2$	0	$\sum_v (\delta - d_v)^-$

e: number of edges of t; d_v: number of children of v; $x = \min(x, 0)$.

$*$: this $1/2$ exponent is classical for series defined through analytic implicit equations.

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3. Compute constants for such trees. . .

Conclusion

- Expansion in the last few years of the study of non-uniform/constraint random permutations (not necessarily in terms of permutons);
  $\rightarrow$ but no big picture at the moment.

- Numerous potential applications:
  - mathematical problems: permutation classes appear naturally in algebraic geometry, dynamical systems, . . . ;
  - in many domains, we have data in the form of permutations (genomics, effective complexity of sorting algorithms).