Combinatorial interpretation and positivity of Kerov's polynomials

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Introduction

- Let us denote by S_n the symmetric group of order n.
- Irreducible representations $\simeq$ partitions $\lambda \vdash n$.
- Normalized character values $\chi_\lambda(\mu)$, for $\mu \in S(n)$?
- Here we are interested in an expression in terms of free cumulants.

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- Here we are interested in an expression in terms of free cumulants.
- Goal : prove that the coefficients are non-negative.
- Tool : a combinatorial formula for character values using maps.

Plan

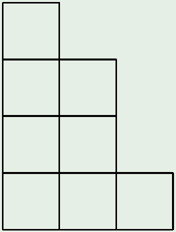
1. Free cumulants and Kerov's polynomials
2. Combinatorial formula for characters
3. Sketch of the proof

Irreducible representations of symmetric groups

- They are indexed by partitions $\lambda \vdash n$, or equivalently by Young diagrams.

Example

- $\lambda_1 = 3; \lambda_2 = \lambda_3 = 2;$
  $\lambda_4 = 1; \lambda_5 = \ldots = 0,$

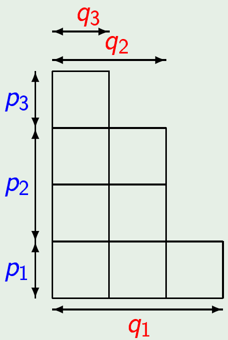


Irreducible representations of symmetric groups

- They are indexed by partitions $\lambda \vdash n$, or equivalently by Young diagrams.
- Other notation : $\lambda = \mathbf{p} \times \mathbf{q}$.

Example

- $\lambda_1 = 3; \lambda_2 = \lambda_3 = 2;$ $\lambda_4 = 1; \lambda_5 = \ldots = 0,$
- $\lambda = (1, 2, 1) \times (3, 2, 1)$



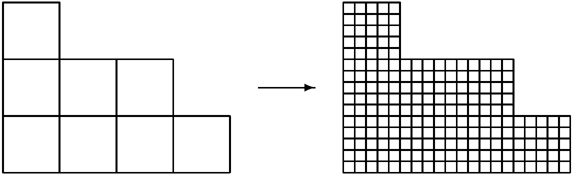
Free cumulants

$$\begin{array}{lll} \text{Young diagram } \lambda & \longrightarrow & \text{Transition measure} \\ & \longrightarrow & \text{Free cumulants } (R_i(\lambda))_{i \geq 2} \end{array}$$

Properties (Biane, 1998)

Homogeneous R_i of degree i in **p** and **q**

Asymptotics $\chi^{\alpha \cdot \lambda}(1 \dots k) \sim_{\alpha \to \infty} R_{k+1}(\lambda) |\alpha \cdot \lambda|^{-(k-1)/2}$



$\lambda = (4, 3, 1)$ $\qquad$ $\alpha \cdot \lambda$

Kerov's polynomials

If $\mu \in S(k) \subset S(n)$ and $\lambda \vdash n$, let

$$\Sigma_\mu^\lambda = n(n-1)\dots(n-k+1)\frac{\chi^\lambda(\mu)}{\chi^\lambda(Id_n)}$$

where χ^λ is the character of the irreducible representation indexed by λ.

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Theorem : Existence of Kerov's polynomials (Kerov, Biane, 2001)

Let $k \geq 1$, there exists a **universal** polynomial K_k such that :

$$\Sigma_{(1\dots k)}^\lambda = K_k(R_2(\lambda), \dots, R_{k+1}(\lambda))$$

K_k does not depend on the diagram λ! $\iff$ equality as power series in $\mathbf{p}$ and $\mathbf{q}$

Description of the coefficients

Asymptotic property of the free cumulants implies:

Proposition

$$K_k = R_{k+1} + \text{ lower degree terms}$$

Moreover :

- K_k has integer coefficients.

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Moreover :

- K_k has integer coefficients.
- We prove here their positivity (conjectured by Kerov and Biane, 2001)

Map of a pair of permutations

pair of permutations $\mapsto$ bicolored edge-labeled map

Example

$$\tau = (14)(253), \qquad \sigma = (13)(254)$$

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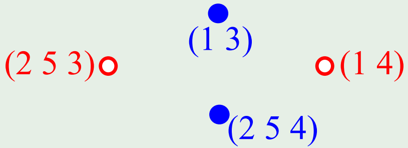

$\circ \quad \leftrightarrow \quad$ cycles of τ

$\bullet \quad \leftrightarrow \quad$ cycles of σ

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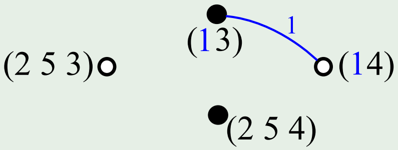

The edge labeled 1 links the two vertices corresponding to cycles containing 1.

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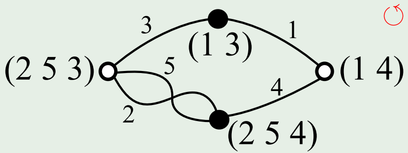

Same thing for the integers between 2 and k. The cyclic order at vertices is given by the cycle on the node.

Map of a pair of permutations

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Example

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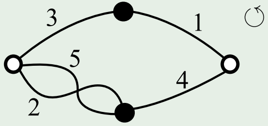


Even if we forget the node labels, we can recover easily the permutations

Colourings of a bicolored map

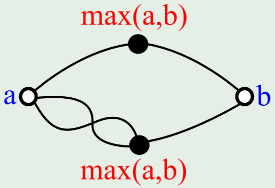
A colouring of the white vertices of M is :

$$\varphi : V_\circ(M) \to \mathbb{N}^\star$$

We associate the following colouring for the black vertices :

$$\psi : \begin{array}{rcl} V_\bullet(M) & \to & \mathbb{N}^\star \\ b & \mapsto & \max\limits_{w \text{ neighbour of } b} \varphi(w) \end{array}$$

Example



Power series associated to a bicolored map

We define the power series in indeterminates **p** and **q** :

$$N(M) = \sum_{\substack{\varphi \text{ colouring of} \\ \text{the white vertices}}} \left(\prod_{w \in V_\circ(M)} p_{\varphi(w)} \prod_{b \in V_\bullet(M)} q_{\psi(b)} \right)$$

$N(M)$ is homogeneous of degree $V_\circ(M) + V_\bullet(M)$ in **p** and **q**!

Example

$$N(M^{\tau,\sigma}) = \sum_{\substack{a \geq 1 \\ b \geq 1}} p_a \cdot p_b \cdot q^2_{\max(a,b)}$$

Combinatorial formulas for character values and cumulants

We will use the following result

Theorem (Stanley, Féray, Śniady, 2006)

With these notations, the character value is:

$$\Sigma_\mu^{\mathbf{p}\times\mathbf{q}} = \sum_{\substack{\tau,\sigma\in S(k)\\ \tau\cdot\sigma=\mu}} (-1)^{|C(\sigma)|+|C(\mu)|} N(M^{\tau,\sigma})(\mathbf{p},\mathbf{q})$$

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The homogeneous component of degree $k+1$ is:

$$R_{k+1}(\mathbf{p}\times\mathbf{q}) = \sum_{\substack{\tau,\sigma\in S(k)\\ \tau\cdot\sigma=(1\ldots k)\\ |C(\tau)|+|C(\sigma)|=k+1}} (-1)^{|C(\sigma)|+1} N(M^{\tau,\sigma})(\mathbf{p},\mathbf{q})$$

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The maps of the pairs of permutations in the second equation are planar trees.

Idea

As power series in $\mathbf{p}$ and $\mathbf{q}$,

$$
\begin{tikzcd}
 & \Sigma_k \arrow[dl, equal, "\text{Kerov polynomial}"'] \arrow[dr, equal, "\text{combinatorial formula for characters}"] & \\
\sum \pm \prod R_{k_i} \arrow[d, equal, "\text{combinatorial formula for cumulants}"'] & & \displaystyle\sum_{M \text{ maps}} \pm N(M) \\
\displaystyle\sum_{F \text{ forests}} \pm N(F) & &
\end{tikzcd}
$$

Idea

As power series in **p** and **q**,

$$\Sigma_k$$

Kerov polynomial

combinatorial formula for characters

$$\sum \pm \prod R_{k_i} \qquad\qquad \sum_{M \text{ maps}} \pm N(M)$$

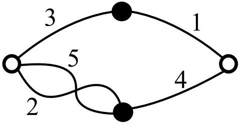
combinatorial formula for cumulants

Can we write each summand $N(M)$ as a sum $\sum \pm N(F)$?

$$\sum_{F \text{ forests}} \pm N(F)$$

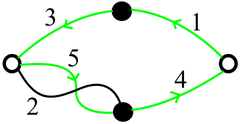
T-transformation

Description on our favorite example



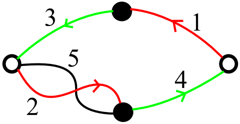
T-transformation

We choose an oriented loop $\vec{L}$ (here dotted)



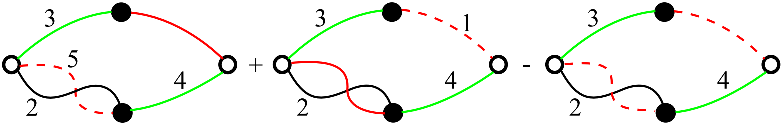
T-transformation

Call erasable its white-to-black directed edges



T-transformation

Let $T_{\vec{L}}(M)$ be the formal expression :



where the dotted edges have been erased.

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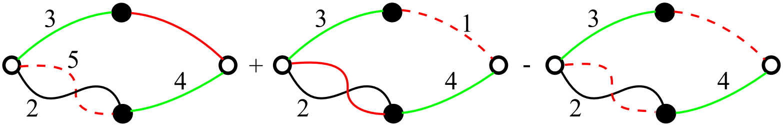

Proposition

$$N\big(T_{\vec{L}}(M)\big) = N(M)$$

Proof. Inclusion/exclusion!

T-transformation

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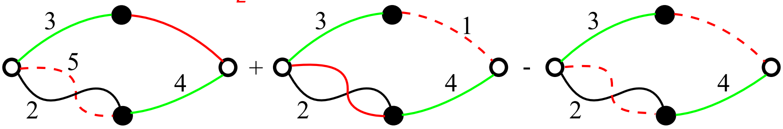

Proposition

$$N\left(T_{\vec{L}}(M)\right) = N(M)$$

Corollary

For any map M, $N(M)$ can be written (not in a unique way!) as

$$N(M) = \sum \pm N(F).$$

Return to our general picture

$$\begin{tikzcd}[column sep=small]
 & \Sigma_k \arrow[dl, equal, "\text{Kerov polynomial}"'] \arrow[dr, equal, "\text{combinatorial formula for characters}"] & \\
\sum \pm \prod R_{k_i} \arrow[d, equal, "\text{combinatorial formula for cumulants}"'] & & \sum_{M \text{ maps}} \pm N(M) \arrow[d, equal, "\text{corollary of the previous slide}"] \\
N\Big(\underbrace{\sum_{F \text{ forests}} \pm F}_{\mathbf{SF}_1}\Big) & & N\Big(\underbrace{\sum_{F \text{ forests}} \pm F}_{\mathbf{SF}_2}\Big)
\end{tikzcd}$$

Return to our general picture

$$\begin{tikzcd}
& \Sigma_k \arrow[dl, equal, "\text{Kerov polynomial}"'] \arrow[dr, equal, "\text{combinatorial formula for characters}"] & \\
\sum \pm \prod R_{k_i} \arrow[d, equal, "\text{combinatorial formula for cumulants}"'] & & \sum\limits_{M \text{ maps}} \pm N(M) \arrow[d, equal, "\text{corollary of the previous slide}"] \\
\underbrace{N\Big(\sum\limits_{F \text{ forests}} \pm F\Big)}_{\mathbf{SF_1}} & & \underbrace{N\Big(\sum\limits_{F \text{ forests}} \pm F\Big)}_{\mathbf{SF_2}}
\end{tikzcd}$$

Question : $\mathbf{SF_1} = \mathbf{SF_2}$? (N is not injective on $\mathbb{Z}$[forests])

Answer

It depends! (there are several ways to write $N(M)$ as $\sum \pm N(F)$)

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Theorem

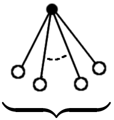
There exists D : $\{$bic. edge-labeled maps$\} \to \mathbb{Z}[$forests$]$ such that :

$$\begin{aligned} N(M) &= N\big(D(M)\big) \\ \mathbf{SF_1} = \mathbf{SF_2} &= \sum_{\substack{\tau,\sigma \in S(k) \\ \tau\cdot\sigma = (12\ldots k)}} (-1)^{|C(\sigma)|+1} D(M^{\tau,\sigma}) \end{aligned}$$

We will explain how to compute D later.

Consequences on Kerov polynomials

Let $T_j =$



$j-1$ white vertices

The combinatorial expression for cumulants give :

$$\prod_i R_{j_i} = \bigsqcup T_{j_i} + \begin{array}{c}\text{forests with at least 1 tree}\\ \text{with more than 1 black vertex}\end{array}$$

$$\begin{array}{c}\text{coefficients}\\ \text{of } \prod R_{j_i}\\ \text{in } K_k\end{array} = \begin{array}{c}\text{coefficients}\\ \text{of } \bigsqcup T_{j_i}\\ \text{in } \mathbf{SF_1}\end{array} = \begin{array}{c}\text{coefficients}\\ \text{of } \bigsqcup T_{j_i}\\ \text{in } \mathbf{SF_2}\end{array}$$

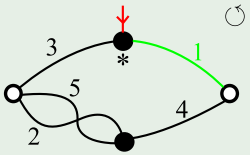
Construction of $D(M)$

1. Add an external half-edge to the map .

   extremity the black extremity $\star$ of the edge e_1 of smallest label

   where in the cyclic order of $\star$? just after e_1.

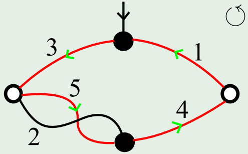
Example



Construction of $D(M)$

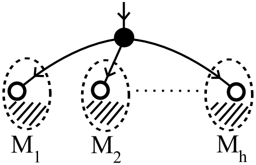
1. Add an external half-edge to the map .
2. Define as admissible the loops :
   - passing through $\star$ if there are any
   - oriented from left to right (it has a sense if we draw the external half-edge on the top of the picture)

Example



Construction of $D(M)$

1. Add an external half-edge to the map .
2. Define as admissible the loops :
    - an admissible loop in one of the M_i (inductive definition)
    - oriented from left to right (it has a sense if we draw the external half-edge on the top of the picture)

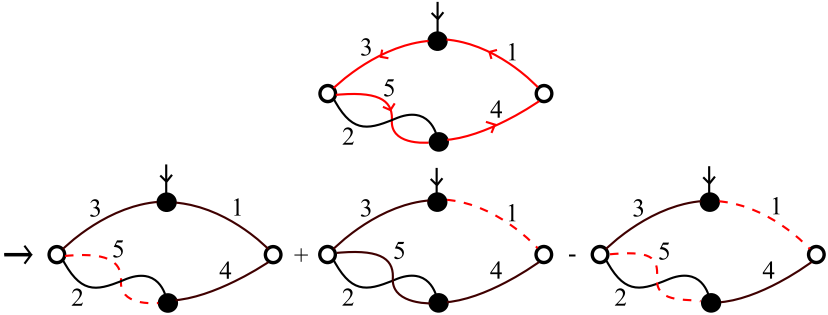


Construction of $D(M)$

1. Add an external half-edge to the map if necessary.
2. Define the admissible loops
3. Apply a $T-$transformation with respect to an admissible loop, without erasing the external half-edge
4. Go back to step 1 with each connected component of each graph of the result.

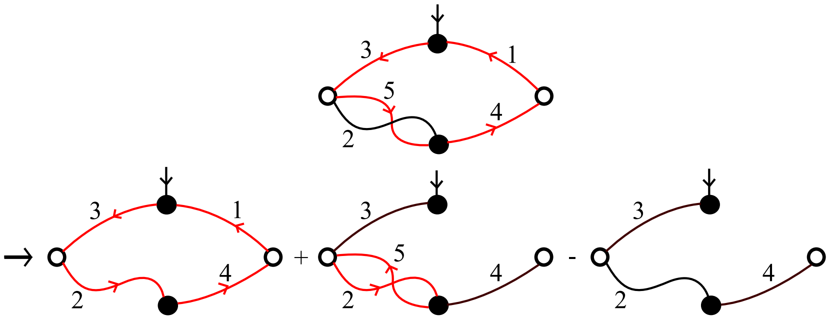
Example

Back to our favorite example :
the loop in the previous example is admissible!



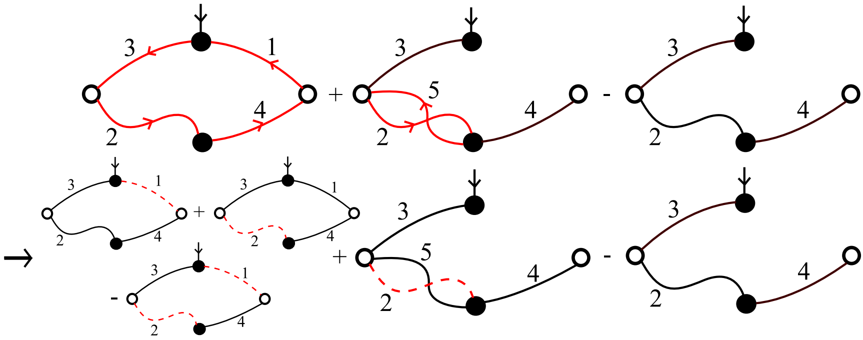
Example

In each one of the resulting maps, there is at most one admissible loop (in red)



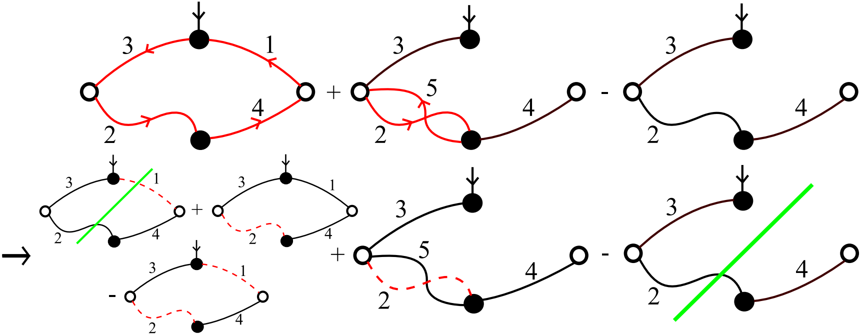
Example

We again apply T-transformations.



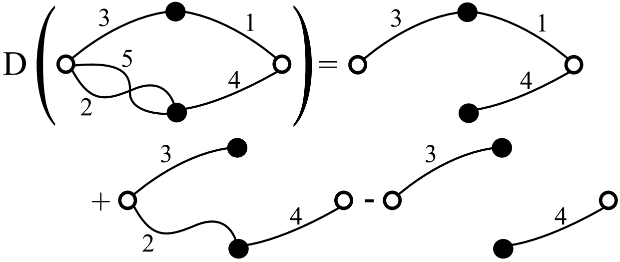
Example

There is a simplification. Note that the trees have coefficient $+1$ and the forest with two components -1.



Example

Final result :



Invariance of the result

There are still some choices to do, but :

Proposition

If we choose only admissible loops, we always obtain the **same** sum of forests denoted $D(M)$.

$D(M)$ has interesting properties :

Proposition

$$N(D(M)) = N(M) \text{(obtained by iterating } T\text{-transformations)}$$

$D(M)$ is an alternate sum of subforests F of M : the sign of the coefficient of F is $(-1)^{\#\text{ c.c. of } F - \#\text{ c.c. of } M}$

D is the decomposition we were looking for!

Theorem

$$\mathbf{SF_1} = \sum_{\substack{\tau,\sigma \in S(k) \\ \tau\cdot\sigma=(12\ldots k)}} (-1)^{|C(\sigma)|+1} D(M^{\tau,\sigma})$$

Sketch of proof :

- We gather terms coming from permutations in a given interval of the symmetric group.
- As intervals are products of non-crossing partions sets, products of free cumulants appear.
- Both sides are decompositions of cumulants.
- Algebraic independance of cumulants finishes the proof.

Proof of Kerov's positivity conjecture

Recall : the coefficient of $\prod_{i=1}^{\ell} Rj_i$ is the coefficient of $\bigsqcup T_{j_i}$ in

$$\mathbf{SF_2} = \sum_{\substack{\tau,\sigma \in S(k) \\ \tau\cdot\sigma=(12\ldots k)}} (-1)^{|C(\sigma)|+1} D(M^{\tau,\sigma})$$

Proof of Kerov's positivity conjecture

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Note that the sum is over connected maps and that the map with a non-zero contribution have ℓ black vertices.

All the contributions have the following sign

$$\underbrace{(-1)^{\ell+1}}_{\text{due to the sign in } \mathbf{SF}_2} \cdot \underbrace{(-1)^{\ell-1}}_{\text{sign of } \bigsqcup T_{j_i} \text{ in } D(M^{\tau,\sigma})} = 1$$

Computation of some coefficients

This method gives more information on coefficients than their positivity :

- We have found a new proof of the compact formula for the highest graduate degree terms in K_k (already computed by I.P. Goulden and A. Ratten and, separately, P. Śniady).
- We can compute the highest degree term in a generalisation about character values on more complex permutations than cycles.

Computation of some coefficients

This method gives more information on coefficients than their positivity :

- Compact expression for highest graduate degree terms.
- We recover the combinatorial interpretation of linear monomials.
- We give a simple combinatorial interpretation for the coefficients of quadratic monomials, which counts permutations.

Computation of some coefficients

This method gives more information on coefficients than their positivity :

- Compact expression for highest graduate degree terms.
- Simple combinatorial interpretations.
- We can give bounds for all the coefficients and link high order cumulants and character values on quite long permutations.

End

Many thanks !,

¡ Gracias !, Merci !

Any questions ?,

¿ Preguntas ?, Questions ?