A determinantal point process approach to random tableaux

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Workshop Cortipom
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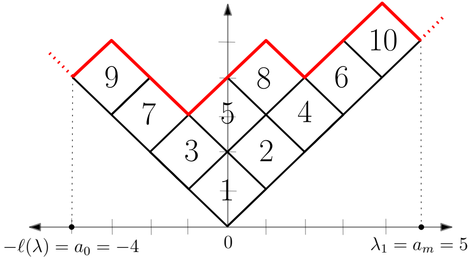
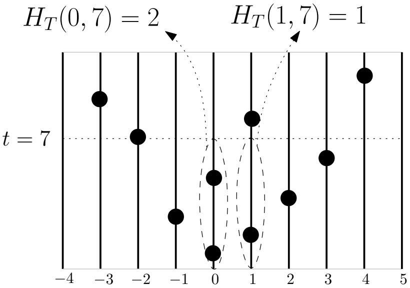
Representation of tableaux as bead configurations





Construction. If $T(x,y) = t$ for some y, then we put a bead on thread x at height t. Then $H_T(x,t)$ is the number of beads below t on thread x.

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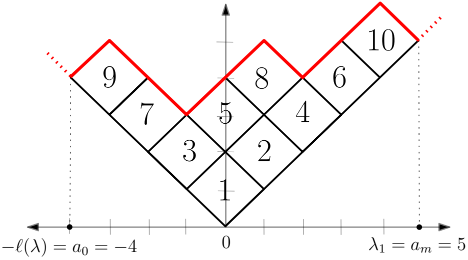

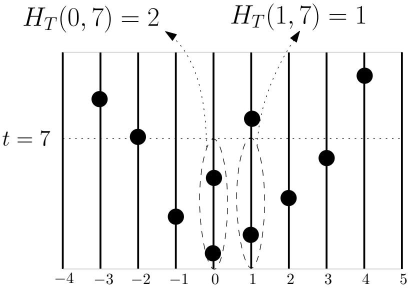

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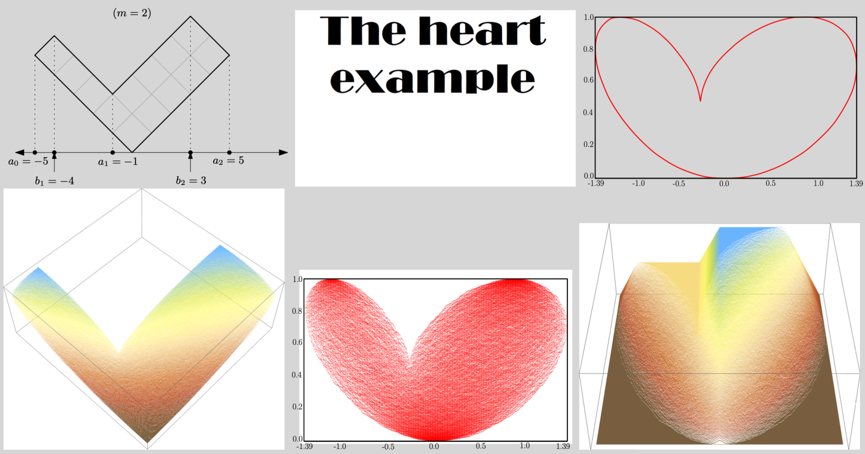
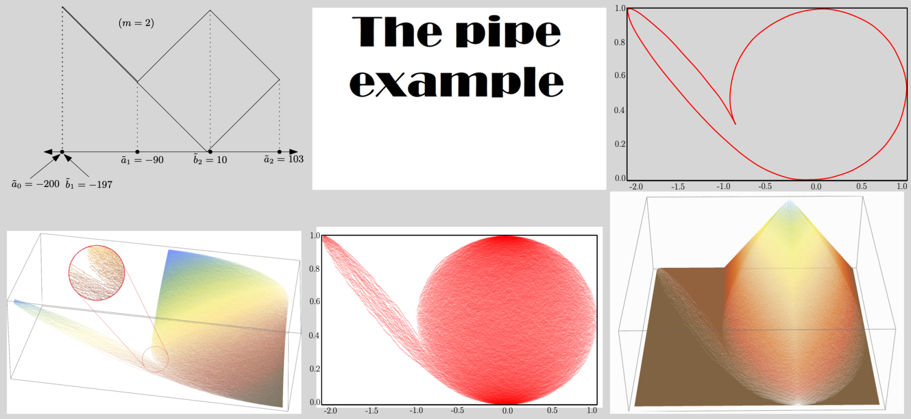
Thm 1 $\leftrightarrow$ asymptotic density of beads at (x,t) is $\frac{1}{(1-t)\pi}\Im(U_c(x,t))$.



Top left: λ^0. Top right: the boundary of the liquid region, i.e. where Thm 1 predicts a positive density of beads.

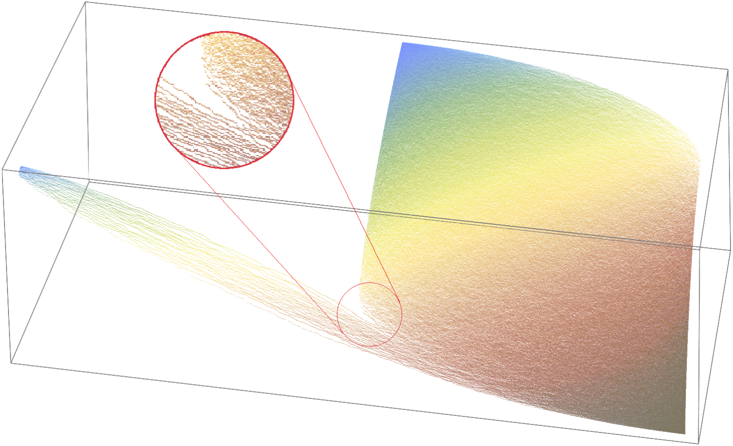
Bottom line: the surface, bead configuration, and height function associated to a uniform random tableau of shape $n \cdot \lambda^0$ for $n = 100$ ($N = 130000$).



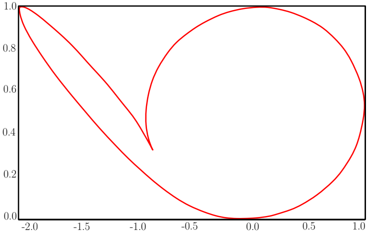
Top left: λ^0. Top right: the boundary of the liquid region, i.e. where Thm 1 predicts a positive density of beads.

Bottom line: the surface, bead configuration, and height function associated to a uniform random tableau of shape $n \cdot \lambda^0$ for $n=6$ ($N=59400$).

Zoom on the pipe example



Why are there discontinuities ?



- There is a discontinuity because for some x, $t \mapsto H(x,t)$ is constant on some interval in the middle, i.e. along a vertical line we have an intermediate zone without beads.

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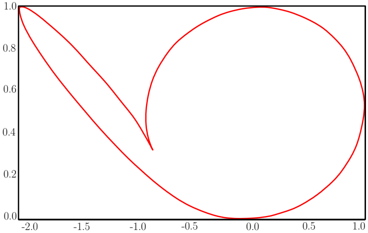

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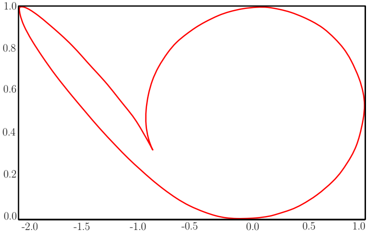

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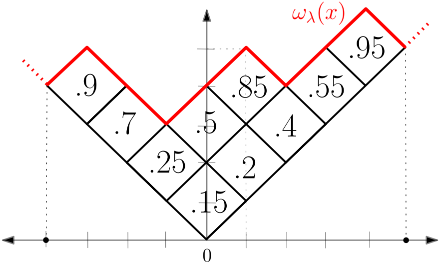
- This happens when the tangent at the cusp is not vertical (both curves at a cusp have the same tangent, think at $y^2 = x^3$).
- To find the criterium for continuity (Theorem 2), it “suffices” to compute the tangent at the cusps.

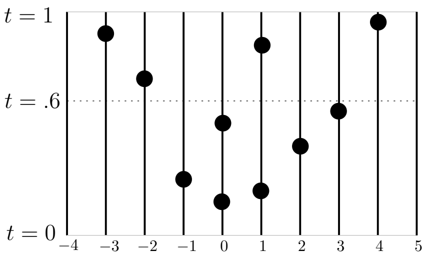
Our main tool: Poissonized tableaux...

A Poissonized tableau T of shape λ is a function $T : \lambda \to \mathbf{[0,1]}$ with the same monotonicity condition as standard tableaux.

Note: the set of Poissonized tableaux is a subset of $[0,1]^\lambda$ with non-empty interior, hence it makes sense to take a uniform random Poissonized tableau of shape λ.

Let M_λ be the associated bead configuration.





. . . and determinantal point processes

Theorem (Gorin, Rahman, '19)

M_λ is a determinantal point process on $\mathbb{Z} \times [0,1]$ with correlation kernel

$$K_\lambda((x_1,t_1),(x_2,t_2)) = \mathbf{1}_{x_1>x_2,\, t_1<t_2} \frac{(t_1-t_2)^{x_1-x_2-1}}{(x_1-x_2-1)!} + \frac{1}{(2i\pi)^2} \oint_{\gamma_z} \oint_{\gamma_w} \frac{F_\lambda(x_2+z)}{F_\lambda(x_1-1-w)} \frac{\Gamma(-w)}{\Gamma(z+1)} \frac{(1-t_2)^z\,(1-t_1)^w}{z+w+x_2-x_1+1}\, dw\, dz,$$

where the double contour integral runs over counterclockwise paths γ_w and γ_z such that

- *γ_w is inside γ_z ;*
- *γ_w and γ_z contain all the integers in $[0, \ell(\lambda)-1+x_1]$ and in $[0, \lambda_1 - 1 - x_2]$ respectively;*
- *the ratio $\frac{1}{z+w+x_2-x_1+1}$ remains uniformly bounded.*

and $F_\lambda(z) := \Gamma(u+1) \prod_{i=1}^{\infty} \frac{u+i}{u-\lambda_i+i}$.

Finding the asymptotics of the kernel

- Let $x_1 = x_2 = x_0\sqrt{N}$, $t_1 = t_2 = t_0$, then the integrand is asymptotically equivalent to

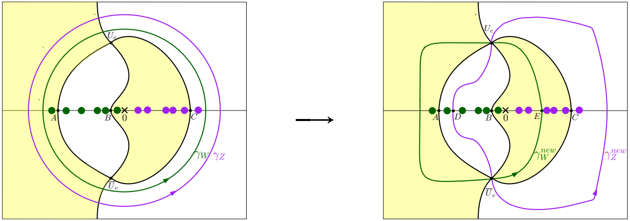
$$\mathrm{Int}_N(W,Z) \simeq N^{-1/2} e^{\sqrt{N}(S(W)-S(Z))} \frac{h(W,Z)}{Z-W},$$

  where
  $S(U) = g(U) - U\log(1-t_0) - \sum_{i=0}^m g(x_0 - \eta\, a_i + U) + \sum_{i=1}^m g(x_0 - \eta\, b_i + U)$;
  $(g(z) = z\log(z))$

This uses essentially Stirling equivalent for the gamma function. . .

Finding the asymptotics of the kernel

- Standard idea: move the contour such that $S(W) - S(Z) < 0$ a.s.



Yellow/white region: small/large values of $\mathrm{Re}\, S(U)$.
The two contours can only meet at a point where $S'(U) = 0$
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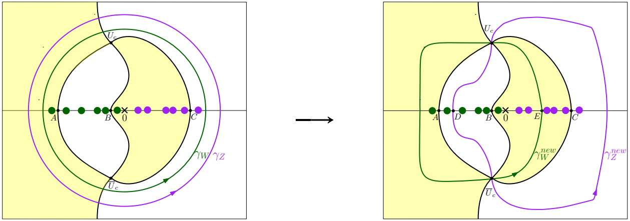

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- Moving the contours yields a residue term of the form: $\frac{N^{-1/2}}{2\pi i} \int_\gamma h(W,W)dW$, where γ goes from $\overline{U_c}$ to U_c. But one can compute $h(W,W) = 1/(1-t_0)$, so that we have

$$K_\lambda((x_0\sqrt{N}, t_0),(x_0\sqrt{N}, t_0)) \simeq \frac{N^{-1/2}}{\pi} \mathrm{Im}(U_c).$$

Discussion

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Thank you for your attention!