An introduction to shifted symmetric functions

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Mini-conférence CORTIPOM,
Paris, juin 2022

Goal of the talk

General introduction on shifted Schur functions:

- analogues of the celebrated Schur functions;
- they satisfy a powerful vanishing property;
- nice extension to the Jack/Macdonald setting.

Shifted symmetric function: definition

Definition

A polynomial $f(x_1, \dots, x_N)$ is shifted symmetric if it is symmetric in $x_1 - 1$, $x_2 - 2$, $\dots$, $x_N - N$.

Example: $p_k^\star(x_1, \dots, x_N) = \sum_{i=1}^{N} (x_i - i)^k$.

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Shifted symmetric function: sequence $f_N(x_1, \ldots, x_N)$ of shifted symmetric polynomials with

$$f_{N+1}(x_1, \ldots, x_N, 0) = f_N(x_1, \ldots, x_N).$$

Example: $p_k^\star = \sum_{i \geq 1} \left[(x_i - i)^k - (-i)^k\right]$.

Shifted Schur functions (Okounkov, Olshanski, '98)

Notation: $\mu = (\mu_1 \geq \cdots \geq \mu_\ell)$ partition.
$(x \downharpoonleft k) := x(x-1)\dots(x-k+1)$;

Definition (Shifted Schur function $s^\star_\mu$)

$$s^\star_\mu(x_1,\dots,x_N) = \frac{\det(x_i + N - i \downharpoonleft \mu_j + N - j)}{\det(x_i + N - i \downharpoonleft N - j)}$$

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Example:

$$s_{(2,1)}(x_1,x_2,x_3) = x_1^2\,x_2 + x_1^2\,x_3 + x_1\,x_2^2 + 2\,x_1\,x_2\,x_3 + x_1\,x_3^2 + x_2^2\,x_3 + x_2\,x_3^2 \\ - x_1\,x_2 - x_1\,x_3 + x_2^2 - x_2\,x_3 + 2\,x_3^2 - 2\,x_2 - 6\,x_3$$

The top degree term of $s^\star_\mu$ is the standard Schur function s_μ.

The vanishing characterization

If λ is a partition (or Young diagram) of length ℓ and F a shifted symmetric function, we denote

$$F(\lambda) := F(\lambda_1, \ldots, \lambda_\ell).$$

Theorem (Vanishing properties of $s^\star_\mu$ (OO '98))

Vanishing characterization $s^\star_\mu$ *is the unique shifted symmetric function of degree at most $|\mu|$ such that $s^\star_\mu(\lambda) = \delta_{\lambda,\mu} H(\lambda)$, where $H(\lambda)$ is the hook product of λ.*

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Extra vanishing property *Moreover,* $s^\star_\mu(\lambda) = 0$*, unless* $\lambda \supseteq \mu$*.*

The vanishing characterization

Proof of the extra-vanishing property.

By definition, $s^{\star}_{\mu}(\lambda) = \frac{\det(\lambda_i+N-i\downharpoonleft\mu_j+N-j)}{\det(\lambda_i+N-i\downharpoonleft N-j)}$.

Call $M_{i,j} = (\lambda_i + N - i \downharpoonleft \mu_j + N - j)$.
If $\lambda_j < \mu_j$ for some j, then $M_{j,j} = 0$,

$$\begin{pmatrix} \ddots & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$

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If $\lambda_j < \mu_j$ for some j, then $M_{j,j} = 0$,
but also all the entries in the bottom left corner.

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Therefore $s^\star_\mu(\lambda) = 0$ as soon as $\lambda \not\supseteq \mu$. □

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To compute $s^\star_\mu(\mu)$, we get a triangular matrix, the determinant is the product of diagonal entries and we recognize the hook product. (Exercise!)

The vanishing characterization

Proof of uniqueness.

Let F be a shifted symmetric function of degree at most $|\mu|$.
Assume that for each λ of size at most μ,

$$F(\lambda) = s^\star_\mu(\lambda) = \delta_{\lambda,\mu} H(\lambda).$$

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Write $G := F - s^\star_\mu$ as linear combination of $s^\star_\nu$:

$$G = \sum_{\nu : |\nu| \leq |\mu|} c_\nu \, s^\star_\nu. \tag{1}$$

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Assume $G \neq 0$, and choose ρ minimal for inclusion such that $c_\rho \neq 0$.
We evaluate (1) in ρ:

$$0 = G(\rho) = \sum_{\nu : |\nu| \le |\mu|} c_\nu \, s^\star_\nu(\rho) = c_\rho \, s^\star_\rho(\rho) \neq 0.$$

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Contradiction $\Rightarrow G = 0$, i.e. $F = s^\star_\mu$. □

A combinatorial formula for $s^\star_\mu$

Theorem (Goulden-Greene '94, OO'98)

$$s^\star_\mu(x_1,\dots,x_N) = \sum_T \prod_{\square\in T} (x_{T(\square)} - c(\square)).$$

where the sum runs over reverse[a] semi-std Young tableaux T, and if $\square = (i,j)$, then $c(\square) = j - i$ (called content).

[a] filling with decreasing columns and weakly decreasing rows

Example:

$$s^\star_{(2,1)}(x_1,x_2) = x_2\,(x_2-1)\,(x_1+1) + x_2\,(x_1-1)\,(x_1+1)$$

2	2
1	

2	1
1	

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[a] filling with decreasing columns and weakly decreasing rows

- extends the classical combinatorial interpretation of Schur function (that we recover by taking top degree terms);
- completely independent proof, via the vanishing theorem (see next slide).

A combinatorial formula for $s^\star_\mu$

To prove: $$s^\star_\mu(x_1,\dots,x_N)=\sum_T \prod_{\square\in T}\left(x_{T(\square)}-c(\square)\right).$$

Sketch of proof via the vanishing characterization.

1. RHS is shifted symmetric:

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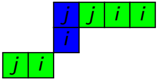
1. RHS is shifted symmetric:

   it is sufficient to check that it is symmetric in $x_i - i$ and $x_{i+1} - i - 1$. Thus we can focus on the boxes containing i and $j := i + 1$ in the tableau and reduce the general case to $\mu = (1,1)$ and $\mu = (k)$. Then it's easy.



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   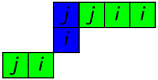

   The compatibility $\mathrm{RHS}(x_1,\dots,x_N,0)=\mathrm{RHS}(x_1,\dots,x_N)$ is straigthforward.

A combinatorial formula for $s^\star_\mu$

$$\text{To prove: } s^\star_\mu(x_1,\dots,x_N) = \sum_T \prod_{\square\in T} (x_{T(\square)} - c(\square)).$$

Sketch of proof via the vanishing characterization.

1. RHS is shifted symmetric: OK.
2. $\mathrm{RHS}\big|_{x_i:=\lambda_i} = 0$ if $\lambda \not\supseteq \mu$.

   We will prove: for each T, some factor $a_\square\big|_{x_i:=\lambda_i} := \lambda_{T(\square)} - c(\square)$ vanishes.

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   - $a_{(1,1)} > 0$;
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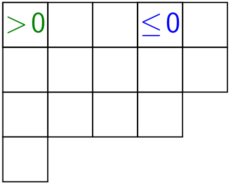

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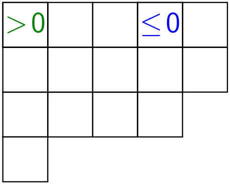

3. Normalization: compare the coefficients of $x_1^{\lambda_1}\dots x_N^{\lambda_N}$. □

A positivity result

We cannot expect positivity in the basis $x_1^{b_1} \dots x_\ell^{b_\ell}$, as for Schur functions, since $s^\star_\mu(\lambda)$ for many partitions λ.

Theorem (Alexandersson, F., '17)

$s^\star_\mu(x_1, \dots, x_n)$ expands with nonnegative rational coefficients in the basis

$$\Big((x_1 - x_2)_{b_1} \cdots (x_{\ell-1} - x_\ell)_{b_\ell - 1} (x_\ell)_{b_\ell} \Big)_{b_1, \dots, b_\ell \geq 0}.$$

Note: it does not follows from the combinatorial interpretation.

Pieri rule for shifted Schur functions

Proposition (OO '98)

$$s^\star_\mu(x_1,\dots,x_N)\,(x_1+\dots+x_N-|\mu|)=\sum_{\nu\colon \nu\nwarrow\mu} s^\star_\nu(x_1,\dots,x_N),$$

where $\nu\nwarrow\mu$ *means* $\nu\supset\mu$ *and* $|\nu|=|\mu|+1$.

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Sketch of proof.

Since the LHS is shifted symmetric of degree $|\mu|+1$, we have

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for some constants c_ν.

- LHS vanishes for $x_i=\lambda_i$ and $|\lambda|\le|\mu| \Rightarrow c_\nu = 0$ if $|\nu|\le|\mu|$. (Same argument as to prove uniqueness.)
- Look at top-degree term (and use Pieri rule for usual Schur functions): $\Rightarrow$ for $|\nu|=|\mu|+1$, we have $c_\nu = \delta_{\nu \nwarrow \mu}$. □

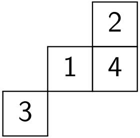
Skew standard tableaux

Definition

Let λ and μ be Young diagrams with $\lambda \subset \mu$. A skew standard tableau of shape λ/μ is a filling of λ/μ with integers from 1 to $r = |\lambda| - |\mu|$ with increasing rows and columns.

Example

$\lambda = (3, 3, 1) \supset \mu = (2, 1)$



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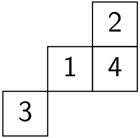

$\leftrightarrow \quad (2,1) \nearrow (2,2) \nearrow (3,2) \nearrow (3,2,1) \nearrow (3,3,1)$

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The number of skew standard tableau of shape λ/μ is denoted $f^{\lambda/\mu}$.

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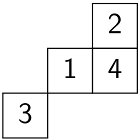

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Skew standard tableaux and shifted Schur functions

Proposition (OO '98)

If $\lambda \supseteq \mu$, then

$$s^\star_\mu(\lambda) = \frac{H(\lambda)}{(|\lambda| - |\mu|)!} \, f^{\lambda/\mu}.$$

Skew standard tableaux and shifted Schur functions

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If $\lambda \supseteq \mu$, then

$$s^\star_\mu(\lambda) = \frac{H(\lambda)}{(|\lambda| - |\mu|)!}\, f^{\lambda/\mu}.$$

Proof.

Set $r = |\lambda| - |\mu|$ We iterate r times the Pieri rule

$$s^\star_\mu(x_1, \dots, x_N)(x_1 + \dots + x_N - |\mu|) \cdots (x_1 + \dots + x_N - |\mu| - r + 1)$$
$$= \sum_{\substack{\nu^{(1)},\dots,\nu^{(r)}: \\ \mu \nearrow \nu^{(1)} \nearrow \dots \nearrow \nu^{(r)}}} s^\star_{\nu^{(r)}}(x_1, \dots, x_N)$$

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$$s^\star_\mu(x_1,\dots,x_N)(x_1+\dots+x_N-|\mu|)\cdots(x_1+\dots+x_N-|\mu|-r+1)$$
$$= \sum_{\substack{\nu^{(1)},\dots,\nu^{(r)}:\\ \mu\nearrow\nu^{(1)}\nearrow\dots\nearrow\nu^{(r)}}} s^\star_{\nu^{(r)}}(x_1,\dots,x_N) = \sum_{\nu:\,|\nu|=|\mu|+r} f^{\nu/\mu} s^\star_\nu(x_1,\dots,x_N).$$

We evaluate at $x_i = \lambda_i$. The only surviving term corresponds to $\nu = \lambda$. □

Shifted Littlewood–Richardson coefficients

Definition

The shifted Littlewood-Richardson coefficients are coefficients $c^{\rho}_{\mu,\nu}$ defined by:

$$s^\star_\mu \, s^\star_\nu = \sum_{\rho : |\rho| \le |\mu| + |\nu|} c^{\rho}_{\mu,\nu} s^\star_\rho$$

Note: when $|\rho| = |\mu| + |\nu|$, then $c^{\rho}_{\mu,\nu}$ is a Littlewood-Richardson coefficient, but $c^{\rho}_{\mu,\nu}$ is defined more generally when $|\rho| < |\mu| + |\nu|$.

Shifted Littlewood-Richardson coefficients

Using the vanishing theorem, one can prove

Proposition (Molev–Sagan '99)

$$c^{\rho}_{\mu,\nu} = \frac{1}{|\rho| - |\nu|} \left(\sum_{\nu^+ \nwarrow \nu} c^{\rho}_{\mu,\nu^+} - \sum_{\rho^- \nearrow \rho} c^{\rho^-}_{\mu,\nu} \right)$$

Allows to compute all $c^{\rho}_{\mu,\nu}$ by induction on $|\rho| - |\nu|$ (μ being fixed).

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Next slide: combinatorial formula for $c^{\rho}_{\mu,\nu}$.
Proof strategy: show that it satisfies the same induction relation.

Shifted Littlewood-Richardson coefficients

Theorem (Molev-Sagan, '99, Molev '09)

$$c^{\rho}_{\mu,\nu} = \sum_{T,R} \mathrm{wt}(T, R),$$

T: reverse semi-standard tableau with barred entries

$\bar{3}$	$\bar{3}$	1	$\bar{1}$
2	$\bar{1}$		
$\bar{1}$			

R: sequence

$$\nu \nearrow \nu^{(1)} \dots \nearrow \nu^{(r)} = \rho.$$

(The barred entries of T indicate in which row is the box $\nu^{(i+1)}/\nu^{(i)}$, so that R is in fact determined by T.)

$$\mathrm{wt}(T, R) := \prod_{\square \text{ unbarred}} \left[\nu^{(k)}_{T(\square)} - c(\square)\right],$$

where $k = \dots$

This contains the usual Littlewood-Richardson rule (only barred entries).

Extensions

Similar theories exist for:

- *P*-Schur functions;
- Jack and Macdonald symmetric functions (see next slides);
- shifted momomial symmetric functions and monomial quasi-symmetric functions.

Problem

Find some deformation of Schur quasi-symmetric functions with nice vanishing properties.

I spend some time on it (with Kelvin Rivera-Lopez), without success. . .

α shifted symmetric functions

Definition

A polynomial $f(x_1, \dots, x_N)$ is α-shifted symmetric if it is symmetric in $x_1 - \frac{1}{\alpha}$, $x_2 - \frac{2}{\alpha}$, ..., $x_N - \frac{N}{\alpha}$.

Examples: $p_k^\star(x_1, \dots, x_N) = \sum_{i=1}^N \left(x_i - \frac{i}{\alpha}\right)^k$.

$\alpha = 1$ gives previous case.

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$\alpha = 1$ gives previous case.

α-shifted symmetric function: sequence $f_N(x_1, \ldots, x_N)$ of shifted symmetric polynomials with

$$f_{N+1}(x_1, \ldots, x_N, 0) = f_N(x_1, \ldots, x_N).$$

Examples: $p_k^\star = \sum_{i \geq 1} \left[(x_i - \frac{i}{\alpha})^k - (\frac{-i}{\alpha})^k \right]$.

Shifted Jack polynomials

Proposition (Sahi, '94)

Let μ be a partition. There exists a unique α-shifted symmetric function $P_\mu^{(\alpha),\star}$ of degree at most $|\mu|$ such that $P_\mu^{(\alpha),\star}(\lambda) = \delta_{\lambda,\mu}\alpha^{-|\mu|}H'_\alpha(\lambda)$ for $|\lambda| \leq |\mu|$.

$H'_\alpha(\lambda)$: deformation of the hook product.

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Note on the proof: looking for $P_\mu^{(\alpha),\star}$ under the form $\sum_{|\nu|\leq|\mu|} c_\nu p_\nu^\star$ the conditions $P_\mu^{(\alpha),\star}(\lambda) = \delta_{\lambda,\mu}H_\alpha(\lambda)$ defines a square system of linear equations in indeterminates c_ν. We need to prove that it is non-degenerate...

Shifted Jack polynomials

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$H'_\alpha(\lambda)$: deformation of the hook product.

Theorem (Knop-Sahi '96, Okounkov '98)

1. $P_\mu^{(\alpha),\star}(\lambda) = 0$ *if* $\lambda \not\supset \mu$ *(extra-vanishing property);*
2. *in general,* $P_\mu^{(\alpha),\star}(\lambda)$ *counts* α*-weighted skew SYT.*
3. *the top degree component of* $P_\mu^{(\alpha),\star}$ *is the usual Jack polynomial* $P_\mu^{(\alpha)}$*.*

$P_\mu^{(\alpha),\star}$ is called shifted Jack polynomials (because of 3.)

No determinantal formula as for shifted Schur functions!...

t shifted symmetric functions

Definition

A polynomial $f(y_1, \dots, y_N)$ is t-shifted symmetric if it is symmetric in $y_1\, t^{-1}$, $y_2\, t^{-2}$, …, $y_N\, t^{-N}$.

Examples: $p_k^\star(y_1, \dots, y_N) = \sum_{i=1}^N \left(y_i\, t^{-i}\right)^k$.

t shifted symmetric functions

Definition

A polynomial $f(y_1, \dots, y_N)$ is t-shifted symmetric if it is symmetric in $y_1\, t^{-1},\ y_2\, t^{-2},\ \dots,\ y_N\, t^{-N}$.

Examples: $p_k^\star(y_1, \dots, y_N) = \sum_{i=1}^{N} \left(y_i\, t^{-i}\right)^k$.

t-shifted symmetric function: sequence $f_N(y_1, \dots, y_N)$ of shifted symmetric polynomials with

$$f_{N+1}(y_1, \dots, y_N, 1) = f_N(y_1, \dots, y_N).$$

Examples: $p_k^\star = \sum_{i \geq 1} \left[(y_i^k - 1)\, t^{-ki}\right]$.

Shifted Macdonald polynomials

Proposition (Sahi '96, Knop '97)

Let μ be a partition. There exists a unique t-shifted symmetric function $P_\mu^{(q,t),\star}$ of degree at most $|\mu|$ such that, for $|\lambda| \leq |\mu|$,

$$P_\mu^{(q,t),\star}(q^{\lambda_1}, q^{\lambda_2}, \dots) = \delta_{\lambda,\mu} H_{(q,t)}(\lambda).$$

$H_{(q,t)}(\lambda)$: deformation of the hook product.

Shifted Macdonald polynomials

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$H_{(q,t)}(\lambda)$: deformation of the hook product.

Theorem (Sahi' 96, Knop '97, Okounkov '98)

1. $P_\mu^{(q,t),\star}(\lambda) = 0$ if $\lambda \not\supset \mu$ (extra-vanishing property);
2. the top degree component of $P_\mu^{(q,t),\star}$ is the usual Macdonald polynomial $P_\mu^{(q,t)}$ evaluated in $y_1, y_2 t^{-1}, \dots, y_n t^{-n}$.

$P_\mu^{(q,t),\star}$ is called shifted Macdonald polynomial.

Note: no interpretation of $P_\mu^{(q,t),\star}(\lambda)$ as counting weighted SYTs!

Positivite expansion in the Jack case

Conjecture (Alexandersson, F., '17)

$P_\mu^{(\alpha),\star}(x_1,\dots,x_n)$ expands with nonnegative rational coefficients in the basis

$$\left(\alpha^a (x_1-x_2)_{b_1}\cdots(x_{\ell-1}-x_\ell)_{b_\ell-1}(x_\ell)_{b_\ell}\right)_{a,b_1,\dots,b_\ell\geq 0}.$$

Theorem (Naqvi, Sahi, Sergel, '21, conjectured by Knop–Sahi. '96)

$(-1)^{|\mu|}H_\mu^{(\alpha)}P_\mu^\star(-x_1-n+1,-x_2-n+2,\dots,-x_n)$ expands with nonnegative integer coefficients as a polynomial in a, x_1, x_2, …, x_{n-1}, x_n.

(In the case $\alpha=1$, Naqvi–Sahi–Sergel theorem follows from the tableau interpretation.)

Shifted Jack-Littlewood Richardson coefficients

Define $c_{\mu,\nu}^{\rho,(\alpha)}$ by

$$P_{\mu}^{(\alpha),\star}\, P_{\nu}^{(\alpha),\star} = \sum_{\rho : |\rho| \leq |\mu|+|\nu|} c_{\mu,\nu}^{\rho,(\alpha)}\, P_{\rho}^{(\alpha),\star}$$

Shifted Jack-Littlewood Richardson coefficients

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$$P_\mu^{(\alpha),\star}\, P_\nu^{(\alpha),\star} = \sum_{\rho:|\rho|\le|\mu|+|\nu|} c_{\mu,\nu}^{\rho,(\alpha)}\, P_\rho^{(\alpha),\star}$$

Conjecture (Alexandersson, F., '19)

$\alpha^{|\mu|+|\nu|-|\rho|-2} H_\alpha(\mu)\, H_\alpha(\nu)\, H'_\alpha(\rho) c_{\mu,\nu}^{\rho,(\alpha)}$ is a polynomial in α with nonnegative integer coefficients.

This implies a conjecture of Stanley ('89, still open), on Jack-Littlewood Richardson coefficients.

Shifted Jack-Littlewood Richardson coefficients

Define $c_{\mu,\nu}^{\rho,(\alpha)}$ by

$$P_\mu^{(\alpha),\star}\, P_\nu^{(\alpha),\star} = \sum_{\rho:|\rho|\leq|\mu|+|\nu|} c_{\mu,\nu}^{\rho,(\alpha)}\, P_\rho^{(\alpha),\star}$$

Conjecture (Alexandersson, F., '19)

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We have an induction relation, as in the Schur case

$$c_{\mu,\nu}^{\rho,(\alpha)} = \frac{1}{|\rho|-|\nu|}\left(\sum_{\nu\nwarrow\nu^+} \psi'_{\nu^+/\nu} c_{\mu,\nu^+}^{\rho,(\alpha)} - \sum_{\rho^-\nearrow\rho} \psi'_{\rho/\rho^-} c_{\mu,\nu}^{\rho^-}\right),$$

but no combinatorial interpretation of this relation.

Thank you for
your attention!