Components of meandric systems and the infinite noodle

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joint work with Paul Thévenin (Upsalla University)

CNRS, Institut Élie Cartan de Lorraine (IECL)

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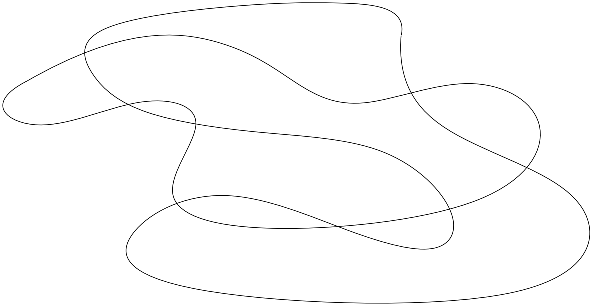
Outline of the talk

A problem in enumerative geometry

We consider two self-avoiding closed curves in the plane crossing generically (no multiple crossing points, no tangeant points).

Problem

How many (non-isomorphic) configurations are there with n intersection points?



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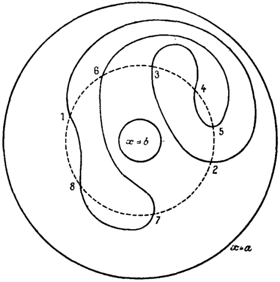
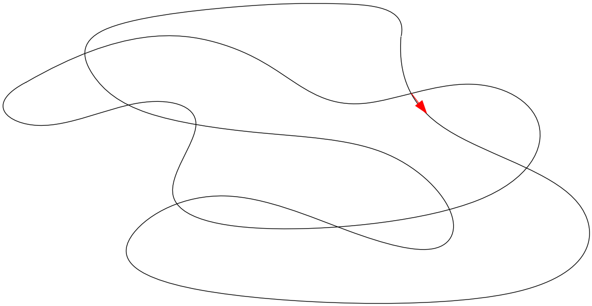


(Fig. 1).

Figure taken from an article of Henri Poincaré, 1912.

Combinatorial reformulation

To avoid symmetries, we root configurations at one intersection point, specifying one of the curve and a direction. The resulting object is called a meander.



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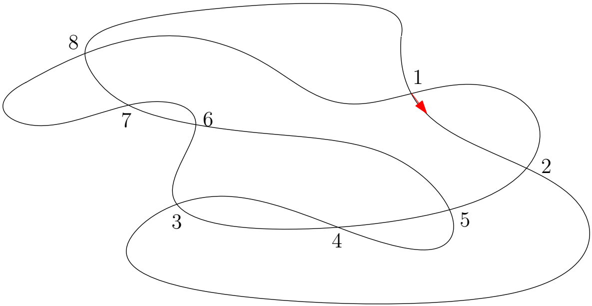

Combinatorial reformulation

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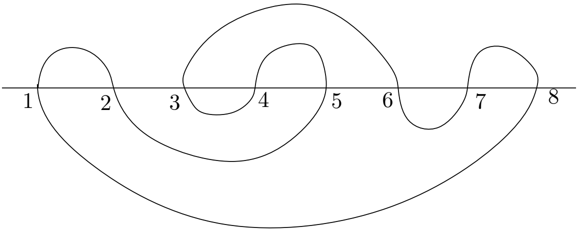
→ we can then label intersection points and transform the marked curve into a straight line.



Combinatorially, a meander is described by two non-crossing pair-partitions, such that the (multi-)graph they induce is connected.

Counting meanders

Let f_n be the number of meanders with n intersection points.

- Easy: $\mathrm{Cat}_n \le f_n \le \mathrm{Cat}_n^2$, where $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$. In particular, f_n grows exponentially.

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- Conjecture (Di Francesco–Golinelli–Guitter, '00): $f_n \sim C\, A^n n^{-\alpha}$, with
$$\alpha = \frac{29+\sqrt{145}}{12}.$$
(Based on theoretical physics heuristics; it matches precisely numerical estimates.)

A related (but easier!) problem

Call meandric system a pair of non-crossing pair-partition and write $\mathrm{cc}(M)$ for its number of components.

Question (Goulden, Nica, Puder, '20)

Let $\boldsymbol{M}_n$ be a uniform random meandric system with n intersection point. What is its average number of components $\mathbb{E}(\mathrm{cc}(\boldsymbol{M}_n))$?

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Theorem (F., Thévenin, '22)

There exists a constant κ (defined in terms of the so-called infinite noodle of Curien–Kozma–Sidoravicius–Tournier) such that $\frac{\mathrm{cc}(\boldsymbol{M}_n)}{n}$ converges to κ in probability. Moreover, $\kappa \in (0.207, 0.292)$.

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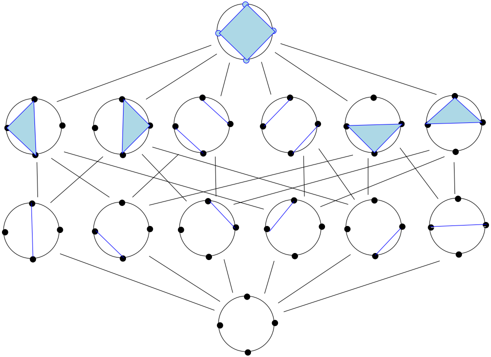
Note: Kargin ('20) suggests that $\mathrm{cc}(\boldsymbol{M}_n)$ is asymptotically normal but we cannot prove this.

Non-crossing partitions

Definition

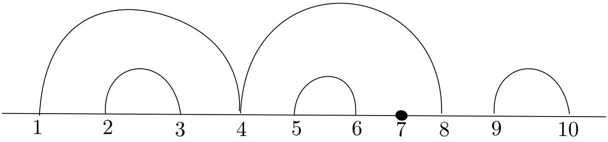
A partition $\pi = \{B_1, \dots, B_r\}$ of the set $\{1, \dots, n\}$ is noncrossing if we cannot find $i < j < k < \ell$ such that i and k are in a block B of π, and j and ℓ in another block $B' \neq B$.



The set $NC(4)$ of noncrossing partitions of $\{1, \dots, 4\}$, ordered by refinement.

From non-crossing partition to non-crossing pair partitions

Consider a non-crossing partition π.



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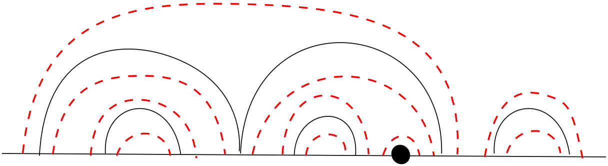

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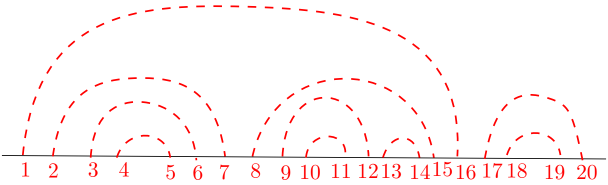

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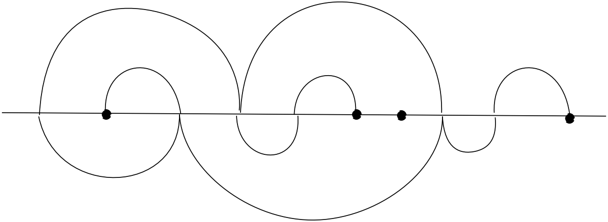
Consider a non-crossing partition π. We associate with it a non-crossing partition. Starting with two non-crossing partitions π, ρ give a meander $M(\pi, \rho)$.



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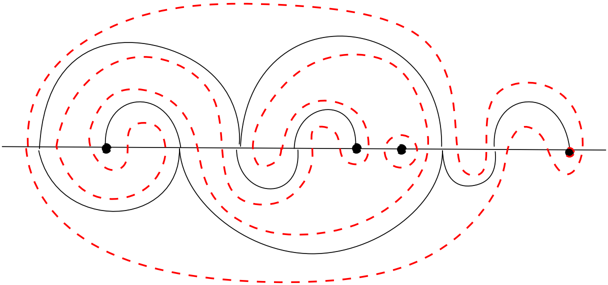

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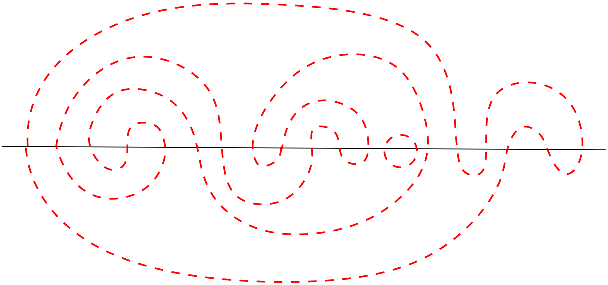

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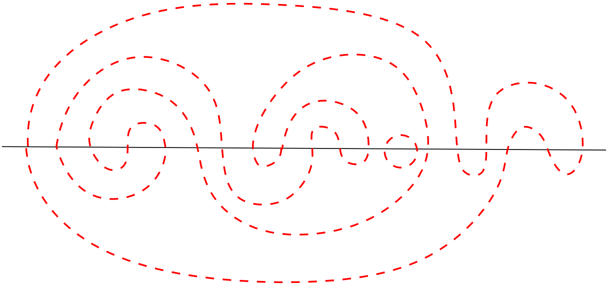

Proposition (Goulden–Nica–Puder, '20)

The graph distance between two noncrossing partitions π and ρ in the Hasse diagram of $NC(n)$ is $n - \mathrm{cc}(M(\pi, \rho))$.

Our main results in terms of noncrossing partitions

We denote d_{H_n} the graph distance in the Hasse diagram of $NC(n)$

Theorem (F., Thévenin, '22)

Let ρ and π be two independent uniform random partitions of n, we have

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Comparison:

- other metric spaces where the 2-point distance concentrates: complete binary tree, d-dimensional hypercube.
- if $\mathbf{0}$ is the partition into singletons, then one can prove $\frac{d(\rho,\mathbf{0})}{n} \to \frac{1}{2}$.
  $\to$ the root is not a typical point (as in the hypercube) and it is not close to geodesics between uniform random points (as in the complete binary tree).

A simple lemma

If M is a meandric system and i in $\{1,\dots,2n\}$, denote $C_i(M)$ the component of M containing i.

Lemma

Let M be a meandric system of size n and $\boldsymbol{i}_n$ a uniform random integer in $\{1,\dots,2n\}$. Then

$$\frac{\mathrm{cc}(M)}{n} = 2 \cdot \mathbb{E}\Big[\frac{1}{|C_{\boldsymbol{i}_n}(M)|}\Big].$$

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Proof.

$\mathbb{E}\Big[\frac{1}{|C_{\boldsymbol{i}_n}(M)|}\Big] = \frac{1}{2n}\left(\sum_{i=1}^{2n} \frac{1}{|C_i(M)|}\right)$, and each component of M contributes 1 to the sum. □

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Consequence: $\frac{1}{n}\mathbb{E}[\mathrm{cc}(M_n)] = 2 \cdot \mathbb{E}\Big[\frac{1}{|C_{\boldsymbol{i}_n}(\boldsymbol{M}_n)|}\Big]$, where $(\boldsymbol{M}_n, \boldsymbol{i}_n)$ is a uniform random "marked meandric system".

Why is the lemma interesting?

Reminder:

$$\frac{1}{n}\mathbb{E}\big[\mathrm{cc}(M_n)\big] = 2 \cdot \mathbb{E}\big[\frac{1}{|C_{\boldsymbol{i}_n}(\boldsymbol{M}_n)|}\big].$$

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The LHS involves a global quantity of M_n (number of components), but the RHS is (most of the time) a local quantity.

Here local means that if we know M_n on a neighbourhood of i_n, we might be able to compute $|C_{\boldsymbol{i}_n}(\boldsymbol{M}_n)|$.

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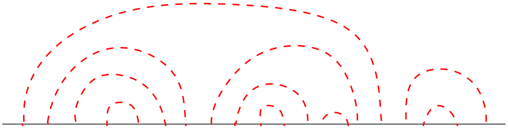
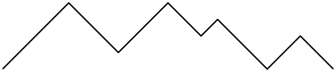
Question

What does M_n look like in the neighbourhood of a random point i_n? In probabilistic words, what is the local limit of $(\boldsymbol{M}_n, \boldsymbol{i}_n)$ (or Benjamini–Schramm limit of $\boldsymbol{M}_n$)?

Path encoding and local limits

Noncrossing pair partitions are in bijection with Dyck paths.





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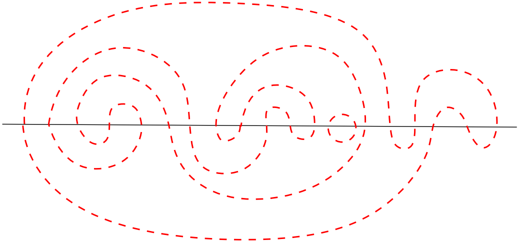

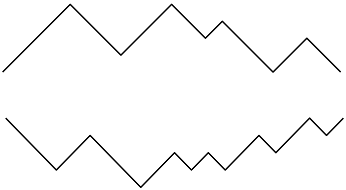

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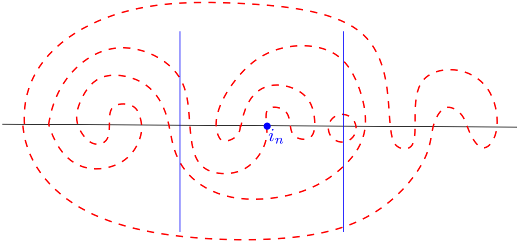
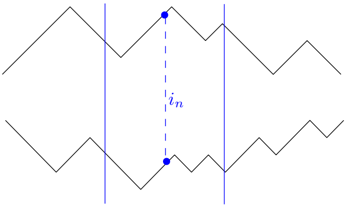
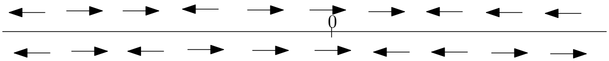




Fact: locally, a uniform random Dyck path “looks like” an unconditioned random walk.

The limiting object (definition)

- Take two bi-infinite sequences of independent uniform random symbols in $\{\rightarrow, \leftarrow\}$;



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- Take two bi-infinite sequences of independent uniform random symbols in $\{\rightarrow, \leftarrow\}$;
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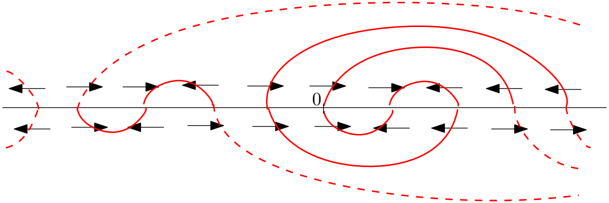

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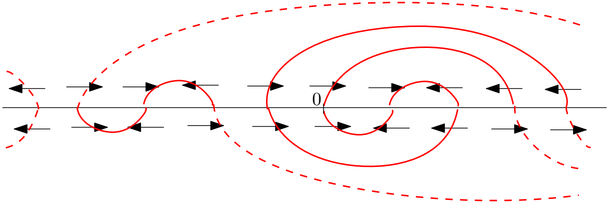

This model has been introduced recently in

[CKST19] N. Curien, G. Kozma, V. Sidoravicius, and L. Tournier. Uniqueness of the infinite noodle, Ann. Inst. Henri Poincaré D, Comb. Phys. Interact. (AIHPD), *6(2):221–238, 2019.*

The limiting object (existence of infinite cluster?)

In [CKST19], the infinite noodle is considered with a percolation point of view, i.e. the authors consider the following question:

Question (CKST, '19)

Are there some infinite clusters in the infinite noodle $\boldsymbol{M}_\infty$?

Easy fact 1 (from ergodic theory): the number $n_\infty(\boldsymbol{M}_\infty)$ of infinite clusters is a.s. constant.

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Easy fact 2: $(n_\infty(\boldsymbol{M}_\infty) = 0 \text{ a.s.}) \Leftrightarrow (C_0(\boldsymbol{M}_\infty) < +\infty \text{ a.s.})$

Local convergence of uniform random meandric system

Proposition (F., Thévenin, '22)

Let $(\boldsymbol{M}_n, \boldsymbol{i}_n)$ be a uniform random "marked meandric system" of size n. Then $(\boldsymbol{M}_n, \boldsymbol{i}_n)$ converges locally in distribution to $(\boldsymbol{M}_\infty, 0)$ in the sense that, for each fixed $R > 0$, the restriction $\boldsymbol{M}_n / [\boldsymbol{i}_n - R, \boldsymbol{i}_n + R]$ converges in distribution to $\boldsymbol{M}_\infty / [-R, R]$.

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Lemma (F., Thévenin, '22)

The functional $(M, r) \mapsto \frac{1}{|C_r(M)|}$ is continuous on the set of complete marked meandric systems.

"Complete" means *without open arcs* (we need to consider meandric systems with open arcs to take restrictions and define the local topology).

Back to the proof of the main theorem

We know that

- $(\boldsymbol{M}_n, \boldsymbol{i}_n)$ converges in distribution to $(\boldsymbol{M}_\infty, 0)$;
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These are bounded r.v., hence they converge also in expectation:

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How to prove convergence in probability and not only in expectation?

- We prove a stronger version of the local convergence, called quenched Benjamini-Schramm convergence.
  In words, we associate to any meandric system M a measure
  $$\mu_M = \frac{1}{2n} \sum_{i=1}^{2n} \delta_{(M,i)}$$
  and we prove the convergence of the random measure $\mu_{\boldsymbol{M}_n}$ to the deterministic measure $\mathrm{Law}((\boldsymbol{M}_\infty, 0))$.

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- We apply a random measure version of the mapping theorem.

Another probabilistic interpretaiton of κ

Lemma

$$\kappa := 2E\left[\frac{1}{|C_0(\boldsymbol{M}_\infty)|}\right] = 2\mathbb{P}[L_0(\boldsymbol{M}_\infty)],$$

where $L_0(\boldsymbol{M}_\infty)$ is the event "0 is the left-most element in its component in $\boldsymbol{M}_\infty$".

Proof.

Conditioning on the size of $C_0(\boldsymbol{M}_\infty)$, by translation invariance, 0 has probability $\frac{1}{|C_0(\boldsymbol{M}_\infty)|}$ to be the left-most element in its component. □

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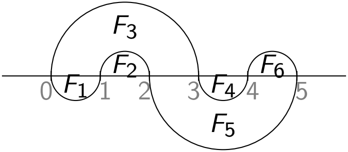
Corollary

Denote $R_0(\boldsymbol{M}_\infty)$ the event "both arrows attached to 0 point to the right". We have

$$\kappa \le 2\,\mathbb{P}\big(R_0(\boldsymbol{M}_\infty)\big) = 0.5.$$

A combinatorial formula for κ (1/2)

For a meander C, we denote $\mathscr{F}(C)$ its set of faces (connected components of the complement):



Proposition (F., Thévenin, '22)

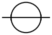
$$\kappa = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{C \in M^{(1),k}} p_C,$$

where $M^{(1),k}$ is the set of meanders of size $2k$ and

$$p_C = 2^{-4k+1} k \sum_{\ell_1,\dots,\ell_{2k-1} \geq 0} \left(\prod_{F \in \mathscr{F}(C)} \mathrm{Cat}_{\ell_{I(F)}} 2^{-2\ell_{I(F)}} \right).$$

Idea: p_C is the probability that $C_0(\boldsymbol{M}_\infty)$ is isomorphic to C.

A combinatorial formula for κ (2/2)

- For $C =$  (which is the only meander of size 2), we have

$$p_C = \frac{1}{8}\sum_{\ell=0}^{\infty} \mathrm{Cat}_\ell^2\, 2^{-4\ell} = \frac{2}{\pi} - \frac{1}{2} \approx 0.137$$

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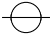
- For $C =$  (which is the only meander of size 2), we have

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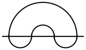
- For $C =$  (which is the only meander of size 4, up to vertical symmetry), we have

$$p_C = \frac{1}{64} \cdot \left(\sum_{\ell_2 \geq 0} \mathrm{Cat}_{\ell_2} \, 2^{-2\ell_2} \right) \cdot \left(\sum_{\ell_1, \ell_3 \geq 0} \mathrm{Cat}_{\ell_1} \, \mathrm{Cat}_{\ell_3} \, \mathrm{Cat}_{\ell_1+\ell_3} \, 2^{-4\ell_1 - 4\ell_3} \right)$$

$$= \frac{1}{64} \cdot 2 \cdot \left(8 - \frac{64}{3\pi} \right) = \frac{1}{4} - \frac{2}{3\pi} \approx 0.038$$

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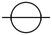
- For $C =$  (which is the only meander of size 2), we have

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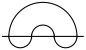
- For $C =$  (which is the only meander of size 4, up to vertical symmetry), we have

$$\begin{aligned} p_C &= \frac{1}{64} \cdot \left(\sum_{\ell_2 \geq 0} \mathrm{Cat}_{\ell_2} \, 2^{-2\ell_2} \right) \cdot \left(\sum_{\ell_1, \ell_3 \geq 0} \mathrm{Cat}_{\ell_1} \mathrm{Cat}_{\ell_3} \mathrm{Cat}_{\ell_1+\ell_3} \, 2^{-4\ell_1 - 4\ell_3} \right) \\ &= \frac{1}{64} \cdot 2 \cdot \left(8 - \frac{64}{3\pi} \right) = \frac{1}{4} - \frac{2}{3\pi} \approx 0.038 \end{aligned}$$

No simple formulas for larger meanders... But we can use the formula to get lower bounds on κ (though it seems to converge slowly).

Question

Find

$$\beta := \lim_{n \to +\infty} \frac{-\log\left(\mathbb{P}(|C_0(\boldsymbol{M}_\infty)| = 2k)\right)}{\log(2k)}$$

I don't even know if $\beta < +\infty$, i.e. if $\mathbb{P}(|C_0(\boldsymbol{M}_\infty)| = 2k)$ decays polynomially fast or not.

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Question (Kargin, '20)

What is the size of the largest component of a uniform random meandric system?

Conjecture (Kargin, '20): $\Theta(n^\alpha)$, with $\alpha \approx 4/5$.

They do not look like standard critical exponents. A naive heuristics (first moment estimates) suggests that $\alpha \beta = 1$.

Thank you for
your attention!