Convergence laws for random permutations

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General problem

Definition

A sequence of random permutations $\sigma_n \in S_n$ satisfies:

- a 0-1 law if, for every first-order property Ψ, the probability $\mathbb{P}[\sigma_n \models \Psi]$ tends to 0 or 1;
- a convergence law if, for every first-order property Ψ, the probability $\mathbb{P}[\sigma_n \models \Psi]$ has a limit as $n \to \infty$?

Motivation: large literature on 0-1/convergence law for random graphs on one side, on random permutations on the other sides.

Goal of today's talk: panorama of some results/questions on the topic.

Permutations as models of a logical theory

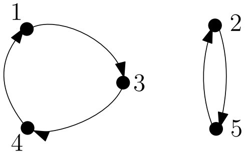
Two ways to see permutations

Bijection point of view

A permutation σ is a pair

$$(A, f),$$

where f a bijection from $A \to A$.



$f(1) = 3$; $f(2) = 5$; $f(3) = 4$;

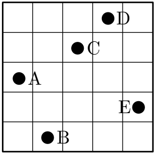
$f(4) = 1$; $f(5) = 2$.

Matrix point of view

A permutation σ is a triple

$$(A, <_P, <_V),$$

where $<_P$ and $<_V$ are linear orders on A.



$A <_P B <_P C <_P D <_P E$;

$B <_V E <_V A <_V C <_V D$.

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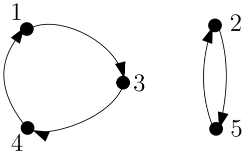

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TOOB: "theory of one bijection"

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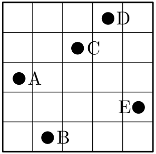

$A <_P B <_P C <_P D <_P E$;
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TOTO: "theory of two orders"

First order properties

Definition: A first order formula is written using variables x, y, z, $\dots$, relational symbols f or $<_P, <_V$, and logical symbols $\exists$, $\forall$, $=$, $\neg$ (we quantify only on variables, not on sets of variables).

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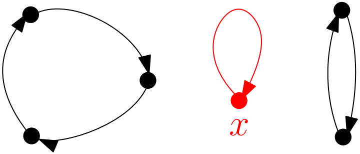
Bijection point of view (TOOB)

Example: existence of a fixed point

$$\exists x : f(x) = x$$



More generally, one can express properties regarding short cycles of the permutation (conjugate permutations are isomorphic!)

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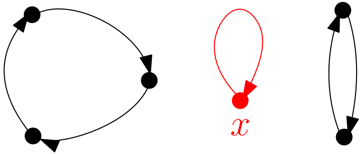

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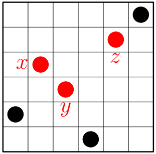
Matrix point of view (TOTO)

Example: existence of a 213 pattern

$$\exists x, y, z : (x <_P y <_P z) \wedge (y <_V x <_P z)$$



More generally, one can express many properties regarding "generalized pattern"… but not the existence of a fixed point!

An "incompatibility" result

Recall that the support of a permutation is the set of its non-fixed points.

Theorem (Albert, Bouvel, F., '20)

Let (P) be a property, expressible as a first-order formula for both TOOB and TOTO. Then

- either all permutations with sufficiently large support verify (P),
- or there is a bound on the size of the support of permutations verifying (P).

Proof uses Ehrenfeucht–Fraïssé games and combinatorial arguments.

Back to 0-1/convergence laws: the bijection point of view

Proposition (folklore?)

Let σ_n a uniform random permutation of n. Then σ_n admits a convergence law for TOOB.

"Proof:" it is well-known that

- σ_n contains a large cycle with high probability;
- the small cycle counts $(\#C_1(\sigma), \#C_2(\sigma), \dots)$ converge jointly to Poisson random variables.

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Theorem (Compton, '87)

Permutations admit an unlabelled 0-1 law. Namely, if σ_n is in the conjugacy class $\mathcal{C}_\lambda$, where λ is taken uniformly at random among partitions of n, then σ_n admits a 0-1 law for TOOB.

(This is one application of a general result, relating 0-1 law and analytic combinatorics.)

Back to 0-1/convergence laws: the matrix point of view

Theorem (Foy, Woods, '90)

Let σ_n be a uniform random permutation. Then σ_n does not admit a convergence law for TOTO (matrix point of view).

In fact, they prove that there exists a first-order property Ψ (using the relations $<_V$, $<_P$) such that

$$\liminf \mathbb{P}(\sigma_n \models \Psi) = 0, \qquad \limsup \mathbb{P}(\sigma_n \models \Psi) = 1.$$

Random pattern avoiding permutation

Definition

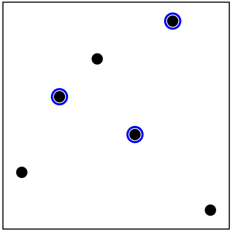
An occurrence of a pattern τ in σ is a subsequence $\sigma_{i_1} \dots \sigma_{i_k}$ that is order-isomorphic to τ, i.e. $\sigma_{i_s} < \sigma_{i_t} \Leftrightarrow \tau_s < \tau_t$.

Example (occurrences of 2 1 3)

2 4 5 3 6 1

8 2 3 4 6 1 7 5

Visual interpretation



We denote $\mathrm{Av}_n(\tau_1, \dots, \tau_r)$ the set of permutations σ of size n avoiding $\tau_1, \dots, \tau_r$. For fixed $\tau_1, \dots, \tau_r$, we consider a uniform random permutation σ_n in $\mathrm{Av}_n(\tau)$.

Convergence law for random avoiding permutations

Theorem (Albert, Bouvel, F., Noy '22)

For each $n \geq 1$, let σ_n be a uniform random permutation in $\mathrm{Av}_n(231)$. Then σ_n satisfies a convergence law.

The proof uses analytic combinatorics; it is based on Woods' approach for convergence law for rooted trees ('97).

Theorem (Braunfeld, Kukla '22)

For each $n \geq 1$, let σ_n be a uniform random permutation in $\mathrm{Av}_n(231, 312)$ (layered permutations). Then σ_n satisfies a convergence law.