On random combinatorial structures: partitions, permutations and asymptotic normality

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Habilitation à diriger les recherches

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My research field(s)

- Algebraic combinatorics: symmetric group representations, symmetric functions, ... with a bias towards formulas suited for asymptotic analysis.
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In my habiliation thesis:

1. Algebraic models of random partitions;
2. Weighted dependency graphs;
3. Universal limits for random pattern-avoiding permutations.

Part 1
Algebraic models of random partitions

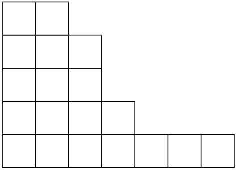
(co-authors: M. Dołęga, P.-L. Méliot)

Partitions

Definition

An integer partition (or *partition* for short) is a non-increasing list $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ of positive integers. Its size is $|\lambda| := \lambda_1 + \cdots + \lambda_\ell$.

It is customary to represent a partition $\lambda = (\lambda_1, \ldots \lambda_\ell)$ by a Young diagram:



$(7,4,3,3,2)$

Plancherel measure

A Young tableaux of shape λ is a filling of λ with numbers from 1 to n with increasing rows and columns.

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4	9	10	15	
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Let $\dim(\lambda)$ be the number of Young tableaux of shape λ. It is the dimension of the *irreducible representation* of S_n associated with λ.

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Definition

The Plancherel measure for Sym_n (or *Plancherel measure* for short) is the probability measure $\mathbb{P}_{\mathrm{Pl}}$ on the set of partitions of n such that for any partition λ of n, we have

$$\mathbb{P}_{\mathrm{Pl}}(\lambda) = \frac{1}{n!}(\dim(\lambda))^2.$$

Well studied since the '60es/'70es (Ulam, Hammersley, Vershik–Kerov, Logan–Shepp, Baik–Deift–Johansson, Borodin–Okounkov–Olshanski, ...)

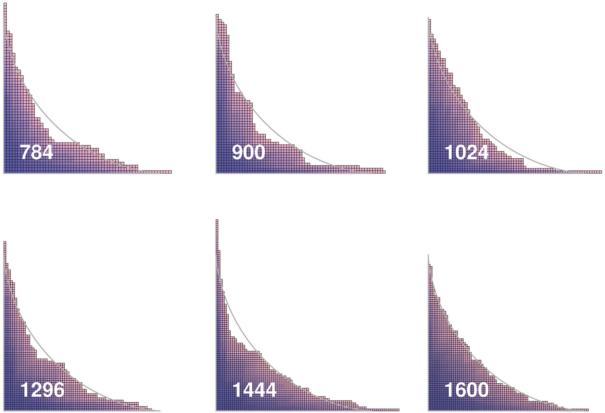
Limit results

Theorem (Logan–Shepp '77/Kerov–Vershik '77)

For each $n \geq 1$, let $\boldsymbol{\lambda}^n$ be a random Young diagram of size n, distributed with Plancherel measure. Then, after rescaling in both directions to have area 1, $\boldsymbol{\lambda}^n$ converges to an explicit limit shape Ω.

A central limit theorem for functionals $F(\boldsymbol{\lambda}^n)$ was given later by Kerov ('93).

Limit results – an illustration

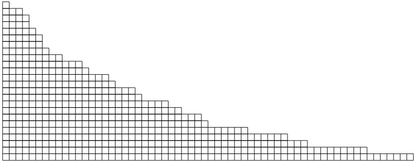


Two generalizations of Plancherel measure

- q-Plancherel measure linked to Hecke algebras (Kerov '92, Strahov '08)

Simulation for $q = 1/2$ (and $n = 200$)

- α-Plancherel measure linked to Jack polynomials (Kerov '00, Fulman '04, ...)



Simulation for $\alpha = 1/3$ (and $n = 500$)

Limit results

Theorem (F. – Méliot '12)

Fix $q < 1$ and let $\boldsymbol{\lambda}^n$ be a random Young diagram of size n, distributed with q-Plancherel measure. Then, for each $i \geq 1$, the quantity $\boldsymbol{\lambda}^n_i / n$ (renormalized length of the i-th row of $\boldsymbol{\lambda}^n$) converges in probability to $(1-q)q^{i-1}$.

Theorem (Dołęga – F. '16)

Fix $\alpha > 0$, and let $\boldsymbol{\lambda}^n$ be a random Young diagram of size n, distributed with α-Plancherel measure. Then, after rescaling rows by a factor $\sqrt{\alpha n}$ and columns by $\sqrt{\frac{n}{\alpha}}$, the diagram $\boldsymbol{\lambda}^n$ converges to the same limit shape Ω as for $\alpha = 1$.

We also have central limit theorems for functionals $F(\boldsymbol{\lambda}^n)$ in both cases.

Proof strategy (following Kerov '93, Ivanov–Olshanski '03)

1. We consider character values of the symmetric group:
$$\overline{\chi}_\sigma(\lambda) := \frac{\mathrm{Tr}[\rho^\lambda(\sigma)]}{\dim(\lambda)}$$
Easy lemma: $\mathbb{E}[\overline{\chi}_\sigma] = 0$ unless $\sigma = \mathrm{id}$.
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2. Express the "shape" of Young diagrams in terms of characters, e.g. for all k, the function $\lambda \mapsto \sum(\lambda_i - i)^k$ is a linear combination of characters. One can find its asymptotics using 1.

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Difficulties: manipulate the analogues of characters in step 1, find the good functionals of the shape for step 2, ... Needed new ideas (e.g. multivariate Stein's method)

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Motto: algebra gives tractable probabilistic models.

Part 2
Weighted dependency graphs

(co-authors: J. Dousse, P.-L. Méliot, A. Nikeghbali)

Context

Consider some sequence of random variables X_n, typically the number of substructures of a given type in a random object.

- number of triangles in random graphs;
- number of exceedances (i s.t. $\sigma(i) \geq i$) in permutations;
- ...

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Consider some sequence of random variables X_n, typically the number of substructures of a given type in a random object.

Goal: prove that X_n is asymptotically normal, i.e., as $n \to +\infty$

$$\frac{X_n - \mathbb{E}[X_n]}{\sqrt{Var(X_n)}} \xrightarrow{d} \mathcal{N}(0,1).$$

Main methods:

- analytic method: Flajolet, Sedgewick, Hwang, ...
- moment/cumulant method: Janson, Mikhailov, ...
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We will present (weighted) dependency graphs, which are based on the moment method.

Dependency graphs

Definition (Malyshev, '80, Petrovskaya/Leontovich, '82, Janson, '88)

A graph L with vertex set A is a dependency graph for the family $\{Y_\alpha, \alpha \in A\}$ if the following holds for any $A_1, A_2 \subset A$:

there is no edge between A_1 and A_2 $\Longrightarrow$ $\{Y_\alpha, \alpha \in A_1\}$ and $\{Y_\alpha, \alpha \in A_2\}$ are independent

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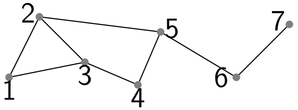
Example

$L =$ 

L is a dependency graph for $\{Y_1, \ldots, Y_7\}$

$\Downarrow$

Y_1 is independent from Y_4, Y_5, Y_6 and Y_7

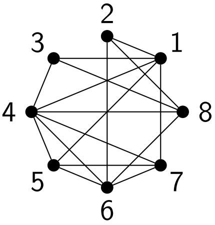
$\{Y_1, Y_2\}$ and $\{Y_4, Y_6, Y_7\}$ are independent

...

Triangles in Erdős-Rényi random graphs

Erdös-Rényi model of random graphs $G(n,p)$:

- G has n vertices labelled $1,\dots,n$;
- each pair $\{i,j\}$ is an edge of G with probability p, and these events are independent from each other.



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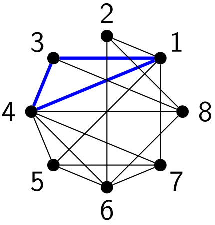

Example of dependency graph

We set $Y_{\{i,j,k\}} = 1$ if G contains the triangle $\{i,j,k\}$ and 0 otherwise. Two Y variables are independent unless the corresponding triangles share an edge. We can encode this in a dependency graph L_n where $\{i,j,k\}$ is linked to $\{i',j',k'\}$ if they have 2 elements (i.e. vertices) in common.

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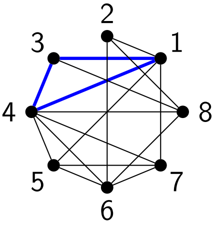

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Claim: L_n is regular with degree $3(n-3)$.

Janson's normality criterion

Setting: for each n,

- let $\{Y_{n,i}, 1 \le i \le N_n\}$ be unif. bounded random variables; $|Y_{n,i}| < M$ a.s.
- we have a dependency graph L_n with maximal degree $D_n - 1$.
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Example: For triangles in $G(n,p)$

$$N_n = \Theta(n^3),\ D_n = \Theta(n) \text{ and } \sigma_n = \Theta(n^2),$$

so that asymptotic normality of the number of triangles follows.

Models with "weak dependencies"

In many models, we do not have independence, but only *weak dependencies*:

1. subword occurrences in a text generated by a Markovian source;
2. subgraph counts in random graphs with fixed number of edges;
3. number of exceedances (i s.t. $\sigma(i) \geq i$) in a uniform random permutation;
4. patterns in multiset permutations and set partitions, . . . ;
5. spins or patterns of spins in Ising model.

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Note: existing theories, such as mixing, work well for models with a spatial structure (1 and 5 in the list); some specific approaches have been developed for 2 and 3.

Weighted dependency graphs

We use weighted graphs, i.e. graphs with a weight in $[0,1]$ on each edge (weight $0 \equiv$ no edge).

Definition (F., '18)

Fix $\boldsymbol{C} = (C_r)_{r \geq 1}$. A weighted graph $\widetilde{L}$ with vertex set A is a **$\boldsymbol{C}$-weighted dependency graph** for the family $\{Y_\alpha, \alpha \in A\}$ if, for any $\alpha_1, \dots, \alpha_r$ in A,

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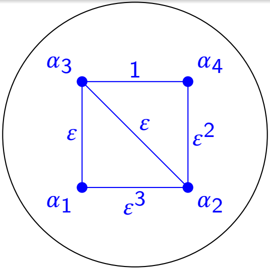

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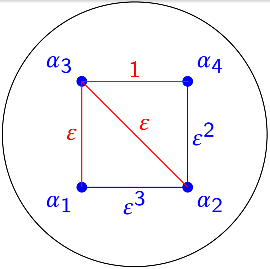
$\widetilde{L}[\alpha_1, \cdots, \alpha_r]$: graph induced by $\widetilde{L}$ on vertices $\alpha_1, \cdots, \alpha_r$.

$\mathcal{M}(K)$: Maximum weight of a spanning tree of K (= product of the edge weights; ε^2 in the example).



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Intuition: the smaller the edge weights are, the smaller the cumulant should be. The edge weights quantify the dependencies between variables.

(Known fact: mixed cumulants of independent r.v. vanish.)

A normality criterion for weighted dependency graphs

Setting: for each n,

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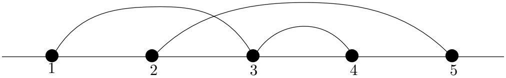
Next few slides: an example of application.

Patterns in set-partitions

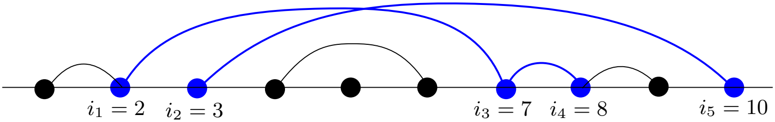
We think at partitions as arch systems, e.g. $\{1,3,4\},\{2,5\}$ is



Definition (Chern, Diaconis, Kane, Rhodes, '14)

An occurrence of a set-partition $\mathscr{A}$ of size ℓ in another set-partition π is a list $(i_1,\dots,i_\ell)$ s.t. (i_j,i_k) is an arch of π whenever (j,k) is an arch of $\mathscr{A}$.

Example: an occurrence of $\{1,3,4\},\{2,5\}$



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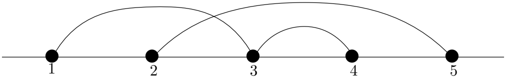

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Background:

- standard well-studied examples: crossings, nestings, k-crossings, k-nestings;
- Chern, Diaconis, Kane and Rhodes proved the asymptotic normality of the number of crossings ('15).

A weighted dependency graph for set-partitions

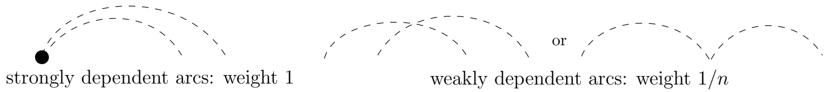
Let $\boldsymbol{\pi}$ be a uniform random set-partition of size n and $\mathbf{1}[\widehat{ij}]$ be the indicator variable of the arc $\{i,j\}$ $(1 \le i < j \le n)$.

Proposition (F., 19)

The complete graph with weights

$$w\left(\mathbf{1}[\widehat{ij}], \mathbf{1}[\widehat{i'j'}]\right) = \begin{cases} 1 & \text{if } i = i' \text{ or } j = j'; \\ 1/n & \text{otherwise.} \end{cases}$$

is a weighted dependency graph for the family $\{\mathbf{1}[\widehat{ij}], i < j\}$.



Asymptotic normality of pattern counts in set partitions

Using a general stability property of weighted-dependency graphs, we get:

Proposition (F., '19)

Fix a pattern $\mathscr{A}$. Let $\mathbf{1}[\pi_I = \mathscr{A}]$ be the indicator of having the pattern $\mathscr{A}$ at position I. Then this family of r.v. has a weighted dependency graph with weights

$$w\big(\mathbf{1}[\pi_I = \mathscr{A}], \mathbf{1}[\pi_{I'} = \mathscr{A}]\big) = \begin{cases} 1 & \text{if } I \cap I' \neq \emptyset; \\ 1/n & \text{otherwise.} \end{cases}$$

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Using (a generalization of) the above normality criterion, we get

Corollary (F., '19, wide generalization of CDKR'14)

For any pattern $\mathscr{A}$, the number $X_n^{\mathscr{A}}$ of occurrences of $\mathscr{A}$ in a uniform random set-partition $\boldsymbol{\pi}$ of $[n]$ is asymptotically normal.

Part 3
Brownian limits for random permutations

(co-authors: F. Bassino, J. Borga, M. Bouvel, M. Drmota
L. Gerin, M. Maazoun, A. Pierrot, B. Stufler)

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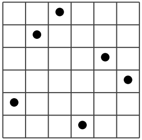
Such *constrained models* are in general very hard. A good situation is when we have a constructive way to describe the objects. Substitution operations may provide such constructive way.

Substitution operation and simple permutations

We see permutations as "diagrams"



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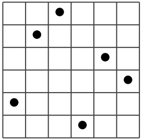

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Substitution operation on permutation

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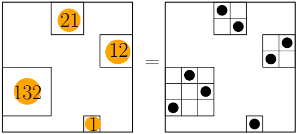

$= 24387156$

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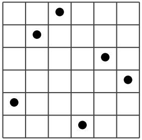

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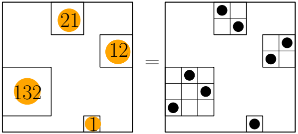

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Definition

A permutation is called simple if it cannot be written as substitution of smaller permutations.

Our problem

- We consider a set of permutations $\mathscr{C}$ defined by the avoidance of some substructures (= patterns) and containing finitely many simple permutations;
- For each n, let $\boldsymbol{\sigma}_n$ be a uniform random permutation of size n in $\mathscr{C}$;
- What is the "limit of the diagram" of $\boldsymbol{\sigma}_n$?

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Note: the limit is in the sense of "permuton"; roughly, we see a permutation as a probability measure on $[0,1]^2$

$$\sigma \ \leftrightarrow \ \frac{1}{n}\sum_{i=1}^{n} \delta_{(i/n,\ \sigma(i)/n)}$$

and we use the notion of weak convergence of measures.

A dichotomy result

Theorem (Bassino, Bouvel, F., Gerin, Maazoun, Pierrot, '19)

In the setting of the previous slide, under an additional technical condition, $\boldsymbol{\sigma}_n$ converges

- *either to a so-called X-permuton;*
- *or to a so-called Brownian separable permuton.*

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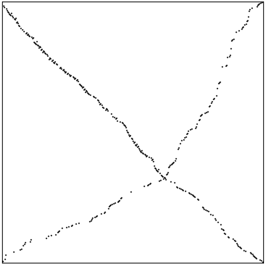

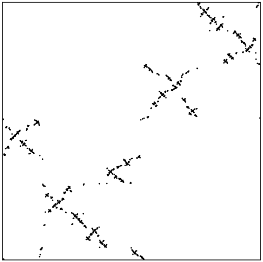

Simulations of large random permutations in classes with finitely many simple permutations.

Some perspectives

1. Analyze algebraic models of random tableaux (= sequences of growing Young diagrams); i.e. we aim at increasing the dimension of the model (adding time).

2. On weighted dependency graphs:
   - more applications: e.g. patterns in conjugacy classes of permutations;
   - example coming from determinantal point processes.

3. Related to random constrained permutations:
   - consider other combinatorial objects (we have some results for graph classes);
   - look for convergence laws for permutations in classes.

That's all folks!

Thank you
for your attention!