An asymptotic bijection and a scaling limit result for fixed genus factorizations of a long cycle

Valentin Féray
joint work with Baptiste Louf and Paul Thévenin

CNRS, Institut Élie Cartan de Lorraine (IECL)

CoRepTiL conference, EPFL
Septembre 2021

Context

We consider

$$\mathscr{F}_n^g = \Big\{ (t_1, \cdots, t_{n-1+2g}) \text{ transpositions in } S_n : t_1 \cdots t_{n-1+2g} = (1\,2 \cdots n) \Big\},$$

i.e. $\mathscr{F}_n^g$ is the set of genus g factorizations of $(1\,2 \cdots n)$ in transpositions.

Context

We consider

$$\mathscr{F}_n^g = \Big\{(t_1, \cdots, t_{n-1+2g}) \text{ transpositions in } S_n : t_1 \cdots t_{n-1+2g} = (1\,2 \cdots n)\Big\},$$

i.e. $\mathscr{F}_n^g$ is the set of genus g factorizations of $(1\,2 \cdots n)$ in transpositions.

Question (Hurwitz, 1891)

Compute $h_{g,n} := |\mathscr{F}_n^g|$.

Remark: $h_{g,n}$ is a particular case of Hurwitz number. It has a more geometric interpretation as the number of genus g covering of the sphere with given types of ramification points (up to isomorphism).

(Asymptotic) enumeration of $\mathscr{F}_n^g$

Dénes ('59) solved the case $g = 0$: $|\mathscr{F}_n^0| = n^{n-2}$ (bijective proofs given later by Moszkowski '89, Goulden–Pepper '93, Goulden–Yong '02, Biane '05).

General case (Jackson '88, Shapiro–Shapiro–Vainshtein '97, Poulhalon–Schaeffer '02):

$$h_{g,n} = \frac{n^{n-2+2g}}{2^{2g}} \sum_{\ell=0}^{g} \binom{n-1+2g}{\ell+2g} \sum_{\substack{\mu \vdash g \\ \ell(\mu)=\ell}} \frac{1}{\mathrm{Aut}(\mu)} \binom{\ell+2g}{2\mu_1+1, \dots, 2\mu_\ell+1}.$$

(Asymptotic) enumeration of $\mathscr{F}_n^g$

Dénes ('59) solved the case $g=0$: $|\mathscr{F}_n^0| = n^{n-2}$ (bijective proofs given later by Moszkowski '89, Goulden–Pepper '93, Goulden–Yong '02, Biane '05).

General case (Jackson '88, Shapiro–Shapiro–Vainshtein '97, Poulhalon–Schaeffer '02):

$$h_{g,n} = \frac{n^{n-2+2g}}{2^{2g}} \sum_{\ell=0}^{g} \binom{n-1+2g}{\ell+2g} \sum_{\substack{\mu \vdash g \\ \ell(\mu)=\ell}} \frac{1}{\mathrm{Aut}(\mu)} \binom{\ell+2g}{2\mu_1+1,\dots,2\mu_\ell+1}.$$

Proofs use representation theory, no combinatorial proof is known!

In particular, for fixed g, as n tends to $+\infty$,

$$h_{g,n} \sim \frac{n^{n-2+5g}}{24^g\, g!}.$$

Main results

For fixed $g > 0$, as n tends to $+\infty$, we obtain:

1. An "asymptotic bijection" proving the asymptotic formula
$$h_{g,n} \sim \frac{n^{n-2+5g}}{24^g g!};$$
2. A scaling limit result for a uniform random element in $\mathscr{F}_n^g$.

Main results

For fixed $g > 0$, as n tends to $+\infty$, we obtain:

1. An "asymptotic bijection" proving the asymptotic formula
$$h_{g,n} \sim \frac{n^{n-2+5g}}{24^g g!};$$
2. A scaling limit result for a uniform random element in $\mathscr{F}_n^g$.

Motivations:

- We need to understand the combinatorial structure (1) in order to analyze random elements (2);
- For (2): there is a large literature on random product of transpositions: independent transpositions, minimal factorizations into adjacent transpositions (sorting networks), ...
- Connections with (random) combinatorial maps;
- An asymptotic bijection could be a first step towards finding a bijection...

Our asymptotic bijection Λ (1/2)

We start with:

- a factorization $F = (t_1, \ldots, t_{n-1+2(g-1)})$ of $(1, \ldots, n)$ genus $g-1$;
- a pair of positions (v, w) in $[1, n-1+2g]$ with $v < w$;
- a triple of values (a, b, c) in $[1, n]$ with $a < b < c$;

Our asymptotic bijection Λ (1/2)

We start with:

- a factorization $F = (t_1, \dots, t_{n-1+2(g-1)})$ of $(1, \dots, n)$ genus $g-1$;
- a pair of positions (v, w) in $[1, n-1+2g]$ with $v < w$;
- a triple of values (a, b, c) in $[1, n]$ with $a < b < c$;

Step 1. We define

$$\overline{F}_1 = (t_1, t_2, \dots, t_{v-1}, (a\,c), t_v, \dots, t_{w-2}, (a\,b), t_{w-1}, \dots, t_{n-1+2(g-1)});$$
$$\overline{F}_2 = (t_1, t_2, \dots, t_{v-1}, (a\,b), t_v, \dots, t_{w-2}, (a\,c), t_{w-1}, \dots, t_{n-1+2(g-1)}).$$

Easy claim: $\overline{F}_1$ and $\overline{F}_2$ are either long cycles or product of three cycles.

Our asymptotic bijection Λ (1/2)

We start with:

- a factorization $F = (t_1, \dots, t_{n-1+2(g-1)})$ of $(1, \dots, n)$ genus $g-1$;
- a pair of positions (v, w) in $[1, n-1+2g]$ with $v < w$;
- a triple of values (a, b, c) in $[1, n]$ with $a < b < c$;

Step 1. We define

$$\overline{F}_1 = (t_1, t_2, \dots, t_{v-1}, (a\,c), t_v, \dots, t_{w-2}, (a\,b), t_{w-1}, \dots, t_{n-1+2(g-1)});$$
$$\overline{F}_2 = (t_1, t_2, \dots, t_{v-1}, (a\,b), t_v, \dots, t_{w-2}, (a\,c), t_{w-1}, \dots, t_{n-1+2(g-1)}).$$

Lemma (F.–Louf–Thévenin, '21)

For almost all (F, v, w, a, b, c), exactly one of $\overline{F}_1$ and $\overline{F}_2$ is a factorization of a long cycle (but not of $(1, \dots, n)$ in general!).

Our asymptotic bijection Λ (2/2)

Step 2. Take the $\overline{F}_i$ which is a factorization of a long cycle, say ζ, and conjugate all transpositions in $\overline{F}_i$ to turn it into a factorization of $(1,\dots,n)$.

Namely, let σ be such that $\sigma(1)=1$ and $\sigma^{-1}\zeta\sigma=(1\cdots n)$ and let $\overline{F}_i=\tau_1,\dots,\tau_{n-1+2g}$;

We set $\Lambda(F,v,w,a,b,c):=(\sigma^{-1}\tau_1\sigma,\dots,\sigma^{-1}\tau_{n-1+2g}\sigma)$.

Our asymptotic bijection Λ (2/2)

Step 2. Take the $\overline{F}_i$ which is a factorization of a long cycle, say ζ, and conjugate all transpositions in $\overline{F}_i$ to turn it into a factorization of $(1,\dots,n)$. Namely, let σ be such that $\sigma(1)=1$ and $\sigma^{-1}\zeta\sigma = (1\cdots n)$ and let $\overline{F}_i = \tau_1,\dots,\tau_{n-1+2g}$;

We set $\Lambda(F,v,w,a,b,c) := (\sigma^{-1}\tau_1\sigma,\dots,\sigma^{-1}\tau_{n-1+2g}\sigma)$.

$\Lambda(F,v,w,a,b,c)$ is a genus g factorization of $(1,\dots,n)$. In other words, Λ maps (almost all) $\mathscr{F}_n^{g-1} \times \binom{[1,n-1+2g]}{2} \times \binom{[1,n]}{3}$ to $\mathscr{F}_n^g$.

Our asymptotic bijection Λ (2/2)

Step 2. Take the $\overline{F}_i$ which is a factorization of a long cycle, say ζ, and conjugate all transpositions in $\overline{F}_i$ to turn it into a factorization of $(1,\dots,n)$. Namely, let σ be such that $\sigma(1)=1$ and $\sigma^{-1}\zeta\sigma=(1\cdots n)$ and let $\overline{F}_i=\tau_1,\dots,\tau_{n-1+2g}$;

We set $\Lambda(F,v,w,a,b,c):=(\sigma^{-1}\tau_1\sigma,\dots,\sigma^{-1}\tau_{n-1+2g}\sigma)$.

$\Lambda(F,v,w,a,b,c)$ is a genus g factorization of $(1,\dots,n)$. In other words, Λ maps (almost all) $\mathscr{F}_n^{g-1}\times\binom{[1,n-1+2g]}{2}\times\binom{[1,n]}{3}$ to $\mathscr{F}_n^g$.

Theorem (F.–Louf–Thévenin, '21)

There exists subsets $\mathscr{A}_{g-1,n}\subset\mathscr{F}_n^{g-1}\times\binom{[1,n-1+2g]}{2}\times\binom{[1,n]}{3}$ and $\mathscr{C}_{g,n}\subset\mathscr{F}_n^g$ of asymptotic proportion 1 such that

$$\Lambda:\mathscr{A}_{g-1,n}\longrightarrow\mathscr{C}_{g,n}$$

is a surjective $2g$-to-1 mapping.

Recovering the asymptotic enumeration of $\mathscr{F}_n^g$

Recall our theorem:

Theorem (F.–Louf–Thévenin, '21)

There exists subsets $\mathscr{A}_{g-1,n} \subset \mathscr{F}_n^{g-1} \times \binom{[1,n-1+2g]}{2} \times \binom{[1,n]}{3}$ *and* $\mathscr{C}_{g,n} \subset \mathscr{F}_n^g$ *of asymptotic proportion* 1 *such that*

$$\Lambda : \mathscr{A}_{g-1,n} \longrightarrow \mathscr{C}_{g,n}$$

is a surjective $2g$*-to-*1 *mapping.*

Recovering the asymptotic enumeration of $\mathscr{F}_n^g$

Recall our theorem:

Theorem (F.–Louf–Thévenin, '21)

There exists subsets $\mathscr{A}_{g-1,n} \subset \mathscr{F}_n^{g-1} \times \binom{[1,n-1+2g]}{2} \times \binom{[1,n]}{3}$ and $\mathscr{C}_{g,n} \subset \mathscr{F}_n^g$ of asymptotic proportion 1 such that

$$\Lambda : \mathscr{A}_{g-1,n} \longrightarrow \mathscr{C}_{g,n}$$

is a surjective $2g$-to-1 mapping.

We have

$$\lim_{n\to+\infty} \frac{|\mathscr{A}_{g-1,n}|}{\frac{n^5}{12}|\mathscr{F}_n^{g-1}|} = 1, \qquad \lim_{n\to+\infty} \frac{|\mathscr{C}_{g,n}|}{|\mathscr{F}_n^g|} = 1, \qquad |\mathscr{A}_{g-1,n}| = 2g\,|\mathscr{C}_{g,n}|,$$

Recovering the asymptotic enumeration of $\mathscr{F}_n^g$

Recall our theorem:

Theorem (F.–Louf–Thévenin, '21)

There exists subsets $\mathscr{A}_{g-1,n} \subset \mathscr{F}_n^{g-1} \times \binom{[1,n-1+2g]}{2} \times \binom{[1,n]}{3}$ *and* $\mathscr{C}_{g,n} \subset \mathscr{F}_n^g$ *of asymptotic proportion* 1 *such that*

$$\Lambda : \mathscr{A}_{g-1,n} \longrightarrow \mathscr{C}_{g,n}$$

is a surjective $2g$*-to-*1 *mapping.*

We have

$$\lim_{n\to+\infty} \frac{|\mathscr{A}_{g-1,n}|}{\frac{n^5}{12}|\mathscr{F}_n^{g-1}|} = 1, \qquad \lim_{n\to+\infty} \frac{|\mathscr{C}_{g,n}|}{|\mathscr{F}_n^g|} = 1, \qquad |\mathscr{A}_{g-1,n}| = 2g\,|\mathscr{C}_{g,n}|,$$

from which we get $\frac{|\mathscr{F}_n^g|}{|\mathscr{F}_n^{g-1}|} \sim \frac{n^5}{24g}$. An easy induction from $|\mathscr{F}_n^0| = n^{n-2}$ gives

$$|\mathscr{F}_n^g| \sim \frac{n^{n-2+5g}}{24^g\, g!}.$$

Encoding factorizations through maps (1/3)

1. Start with the following factorization in $\mathscr{F}_9^2$:

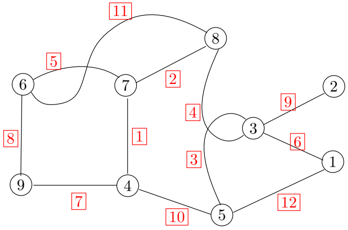
$$(47)(78)(35)(38)(67)(13)(49)(69)(23)(45)(68)(15).$$

Encoding factorizations through maps (1/3)

1. Start with the following factorization in $\mathscr{F}_9^2$:

$$(47)(78)(35)(38)(67)(13)(49)(69)(23)(45)(68)(15).$$

2. For each transposition $\tau_i = (j,k)$, we draw an edge $\{j,k\}$ with label i. (Around each vertex, edges are oriented counterclockwise in increasing order of their labels.)



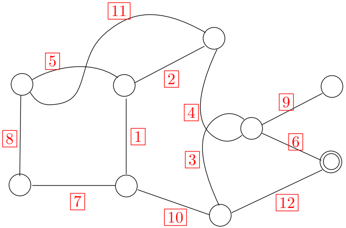
Encoding factorizations through maps (1/3)

1. Start with the following factorization in $\mathscr{F}_9^2$:

$$(47)(78)(35)(38)(67)(13)(49)(69)(23)(45)(68)(15).$$

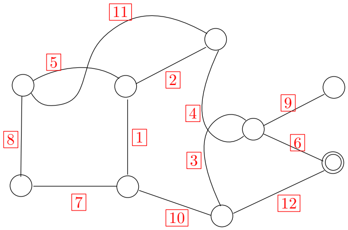
2. For each transposition $\tau_i = (j,k)$, we draw an edge $\{j,k\}$ with label i. (Around each vertex, edges are oriented counterclockwise in increasing order of their labels.)

3. Root the map at vertex 1 and forget vertex labels.

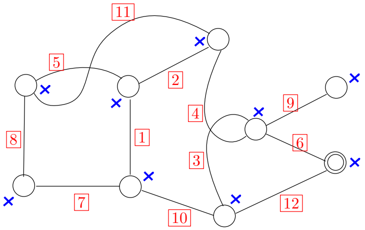


Encoding factorizations through maps (2/3)

Claim: we can recover the vertex labels (and hence the factorization) from the edge labels.



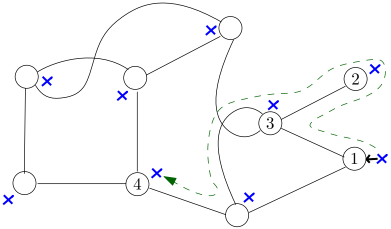
Claim: we can recover the vertex labels (and hence the factorization) from the edge labels.



1. Find the corner of each vertex which is between the incident edges of minimal and maximal labels (called special corner).

Encoding factorizations through maps (2/3)

Claim: we can recover the vertex labels (and hence the factorization) from the edge labels.



1. Find the corner of each vertex which is between the incident edges of minimal and maximal labels (called special corner).
2. Start at the special corner of the root, turn around the map and label vertices from 1 to n in increasing order when crossing their special corner.

NB: to label all vertices, the map must be unicellular!

Encoding factorizations through maps (3/3)

Definition

A Hurwitz map is an edge-labelled map such that around each vertex, edges are oriented counterclockwise in increasing order of their labels (Hurwitz condition).

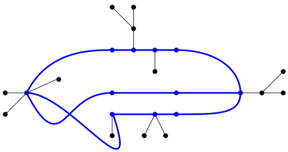
Theorem (Irving, '04)

The construction in the previous slide is a bijection from $\mathscr{F}_n^g$ to the set $\mathscr{H}_n^g$ of vertex-rooted unicellular Hurwitz maps with n vertices and genus g.

Typical structure of unicellular (Hurwitz) maps

Lemma

Take g fixed and n large. Typically, most vertices of a unicellular (Hurwitz) maps in $\mathscr{H}_n^g$ are outside its 2-core.

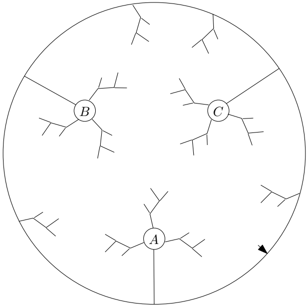


The 2-core of a map

Proof by analytic combinatorics: one can decompose maps (with a marked vertex) as a skeleton where we attach trees...

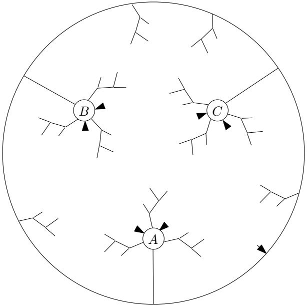
Adding 2 edges to increase the genus

Here is a schematic representation of a Hurwitz map with three marked vertices: the outer circle is the 2-core which has been unfolded (recall that the map is unicellular).



Adding 2 edges to increase the genus

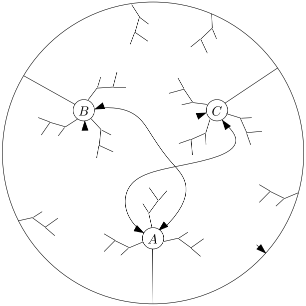
Here is a schematic representation of a Hurwitz map with three marked vertices: the outer circle is the 2-core which has been unfolded (recall that the map is unicellular).

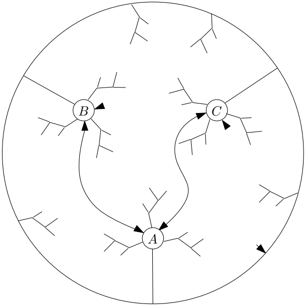


We want to add two edges $\{A; B\}$ and $\{A, C\}$ with labels v and w between these three vertices. Because of the Hurwitz condition, we need to add them at specific corners of A, B and C (we do not know which corner corresponds to v, and which to w!).

Adding 2 edges to increase the genus

Here is a schematic representation of a Hurwitz map with three marked vertices: the outer circle is the 2-core which has been unfolded (recall that the map is unicellular).

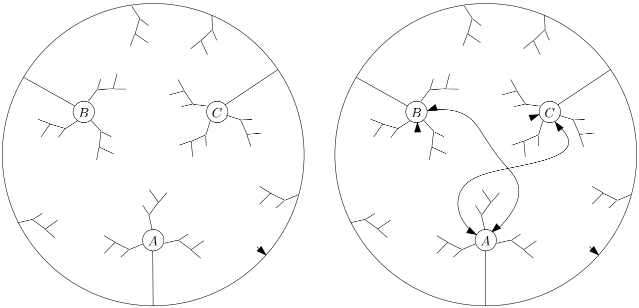




There are two possibilities: one gives a unicellular map, one does not.

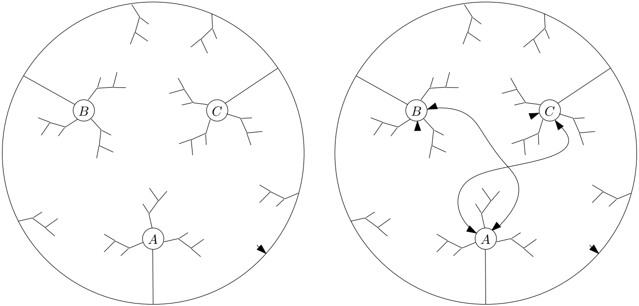
$\longrightarrow$ exactly one of $\overline{F}_1$ and $\overline{F}_2$ is a factorization of a long cycle.

Back to permutations



- Adding two edges $\{A;B\}$ and $\{A,C\}$ with labels v and w in the map corresponds in adding the transposition $(a\,b)$ and $(a\,c)$ at positions v and w;

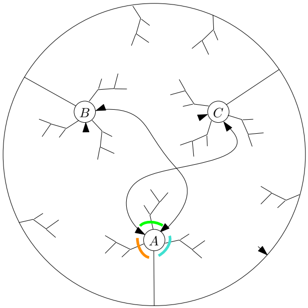
Back to permutations



- Adding two edges $\{A; B\}$ and $\{A, C\}$ with labels v and w in the map corresponds in adding the transposition $(a\,b)$ and $(a\,c)$ at positions v and w;
- The contour order of the face is changing. Hence, the vertex labels are changing, which explain the conjugation in our asymptotic bijection on factorizations.

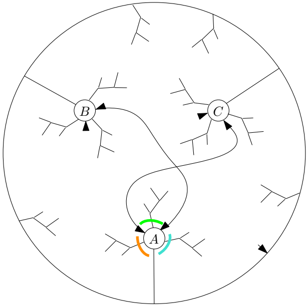
How to invert the construction? (1/2)

Step 1: identify A. When turning around the unique face of the map, the corners of A are not visited in counterclockwise order (in our picture, they are visited in the blue-red-green order). Such a vertex is called a trisection (Chapuy, '09).



How to invert the construction? (2/2)

When A is identified, it is typically easy to know which edges have been added (edges adjacent to A that belong to the 2-core, but not the first visited one).



How to invert the construction? (2/2)

When A is identified, it is typically easy to know which edges have been added (edges adjacent to A that belong to the 2-core, but not the first visited one).

$\to$ the number of pre-image is typically the number of trisections.

How to invert the construction? (2/2)

When A is identified, it is typically easy to know which edges have been added (edges adjacent to A that belong to the 2-core, but not the first visited one).

→ the number of pre-image is typically the number of trisections.

Lemma (Chapuy, '09)

The number of trisections[a] *in a unicellular map is* $2g$.

[a] Counted with multiplicities but, typically there is no multiplicities.

→ this explains why the map Λ is typically a $2g$-to-1 correspondance.

First application: sampling

Let F be a (random) factorization in $\mathscr{F}_n^{g-1}$. We set

$$\mathbf{\Lambda}(F) = \Lambda(F, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}),$$

where $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are taken uniformly at random.

Proposition (F.–Louf–Thévenin, '21)

Let $\boldsymbol{F}_n^0$ and $\boldsymbol{F}_n^g$ be uniform random factorizations of $(1, \ldots, n)$ of genera 0 and g, respectively. Then

$$\lim_{n \to \infty} d_{TV}\big(\mathbf{\Lambda}^g(\boldsymbol{F}_n^0), \boldsymbol{F}_n^g\big) = 0.$$

Reminder: total variation distance between random variables taking values in a discrete set S

$$d_{TV}(X, Y) = \frac{1}{2} \sum_{k \in S} |\mathbb{P}[X = k] - \mathbb{P}[Y = k]|.$$

First application: sampling

Let F be a (random) factorization in $\mathscr{F}_n^{g-1}$. We set

$$\boldsymbol{\Lambda}(F) = \Lambda(F, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}),$$

where $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are taken uniformly at random.

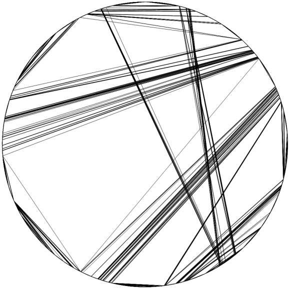
Proposition (F.–Louf–Thévenin, '21)

Let $\boldsymbol{F}_n^0$ and $\boldsymbol{F}_n^g$ be uniform random factorizations of $(1, \dots, n)$ of genera 0 and g, respectively. Then

$$\lim_{n \to \infty} d_{TV}\big(\boldsymbol{\Lambda}^g(\boldsymbol{F}_n^0), \boldsymbol{F}_n^g\big) = 0.$$

$\boldsymbol{F}_n^0$ is easy to sample in linear time. The proposition gives an algorithm to sample a asymptotically uniform genus g factorization in linear time.

Simulation

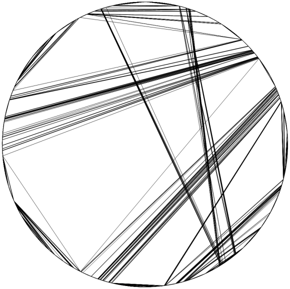


A random factorization of size $n = 1000$
and genus $g = 1$ generated by our algorithm.

Here a transposition (a, b) is encoded by a chord

$$[\exp(-2\pi i\, a/n), \exp(-2\pi i\, b/n)].$$

Simulation



A random factorization of size $n = 1000$
and genus $g = 1$ generated by our algorithm.

Here a transposition (a, b) is encoded by a chord

$$[\exp(-2\pi i\, a/n), \exp(-2\pi i\, b/n)].$$

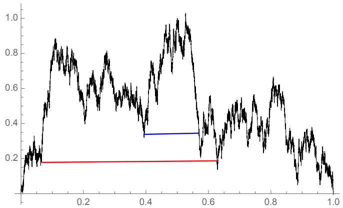
Question

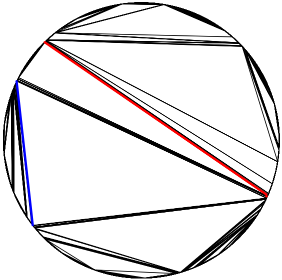
What is the scaling limit of (the set of chords corresponding to) $\boldsymbol{F}_n^g$?

Theorem (F., Kortchemski, '18, Thévenin, '21)

The set of chords associated to $\boldsymbol{F}_n^0$ converges in distribution to Aldous' Brownian triangulation (for the Hausdorff distance on compact subsets of the disk).

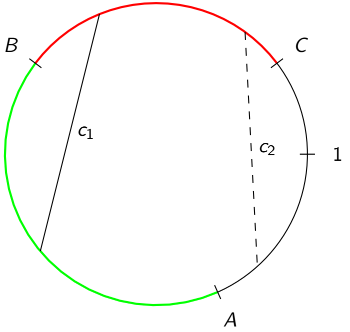
What is Aldous' Brownian triangulation?
Start from a Brownian excursion $\mathbb{X}_\infty$ and draw a chord $[e^{-2\pi i s}, e^{-2\pi i t}]$ for each tunnel (s, t) in $\mathbb{X}_\infty$.

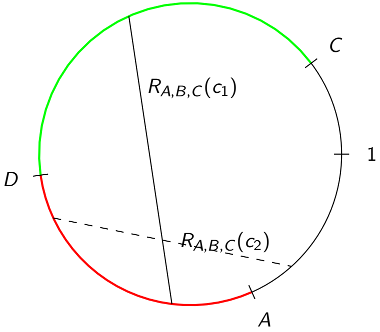




Scaling limit in higher genus $g > 0$ (1/2)

We introduce a rotation operation on (set of) chords. Informally, given three points A, B, C on the circle, $R_{A,B,C}$ "swaps" the arcs of circles AB and BC.



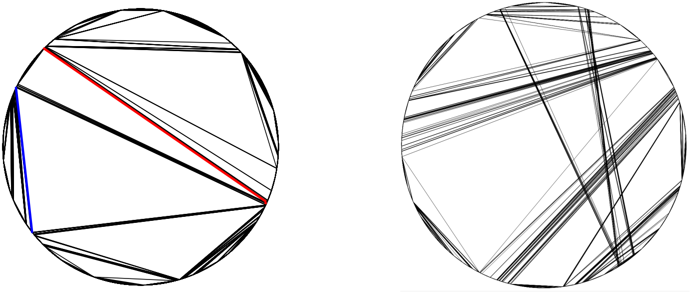


Taking $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ uniformly at random, we denote $\boldsymbol{R}$ the corresponding rotation.

Scaling limit in higher genus $g > 0$ (2/2)

Theorem (F., Louf, Thévenin, '21)

The set of chords associated with $\boldsymbol{F}_n^g$ converges in distribution to $\overline{\boldsymbol{R}^g(\mathbb{L}_\infty)}$, where $\mathbb{L}_\infty$ is Aldous' Brownian triangulation.

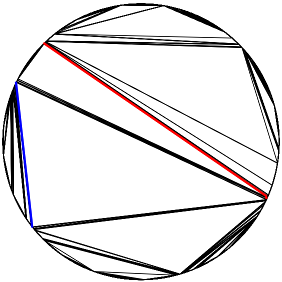


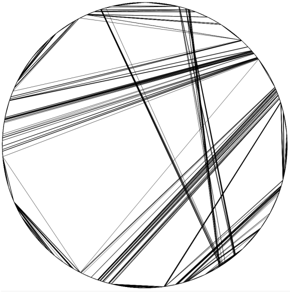
(Left) Brownian triangulation; (Right) a realization of $\boldsymbol{F}_{1000}^1$.

Scaling limit in higher genus $g > 0$ (2/2)

Theorem (F., Louf, Thévenin, '21)

The set of chords associated with $\boldsymbol{F}_n^g$ converges in distribution to $\overline{\boldsymbol{R}^g(\mathbb{L}_\infty)}$, where $\mathbb{L}_\infty$ is Aldous' Brownian triangulation.





Note: we have a "process version" of the result of – where chords are added one at the time, in the order in which they appear in the factorization.

Proof strategy

1. We can replace $\boldsymbol{F}_n^g$ by $\boldsymbol{\Lambda}^g(\boldsymbol{F}_n^0)$; since $\boldsymbol{F}_n^0$ converges to the Brownian triangulation, we need to understand the "effect" of Λ.

Proof strategy

1. We can replace $\boldsymbol{F}_n^g$ by $\boldsymbol{\Lambda}^g(\boldsymbol{F}_n^0)$; since $\boldsymbol{F}_n^0$ converges to the Brownian triangulation, we need to understand the "effect" of Λ.

2. Recall that Λ is defined in two steps:
i) adding transpositions (a, b) and (a, c) to the transpotition;
ii) conjugating all transpositions by some permutation σ.

Proof strategy

1. We can replace $\boldsymbol{F}_n^g$ by $\boldsymbol{\Lambda}^g(\boldsymbol{F}_n^0)$; since $\boldsymbol{F}_n^0$ converges to the Brownian triangulation, we need to understand the "effect" of Λ.

2. Recall that Λ is defined in two steps:
i) adding transpositions (a,b) and (a,c) to the transpotition;
ii) conjugating all transpositions by some permutation σ.

Lemma (see next slide for a heuristics)

Define

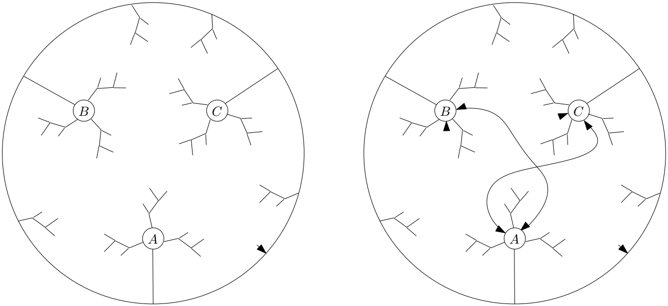
$$\tilde{\sigma}(j) = \begin{cases} j & \text{if } j \leq a \text{ or } j > c; \\ j+c-b & \text{if } a < j \leq b; \\ j-b+a & \text{if } b < j \leq c. \end{cases}$$

Then $\sigma(j) = \tilde{\sigma}(j) + \mathcal{O}_P(1)$, except for j in a set of size $\mathcal{O}_P(1)$.

⇒ Conjugating a transposition by σ acts as $R_{A,B,C}$ on the associated chord.

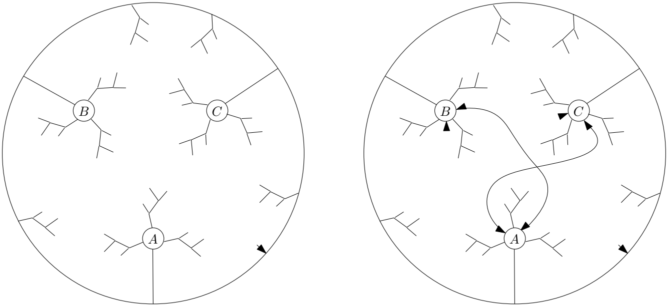
The relabeling permutation σ

On Hurwitz maps, our asymptotic bijection consists in adding two edges;



The relabeling permutation σ

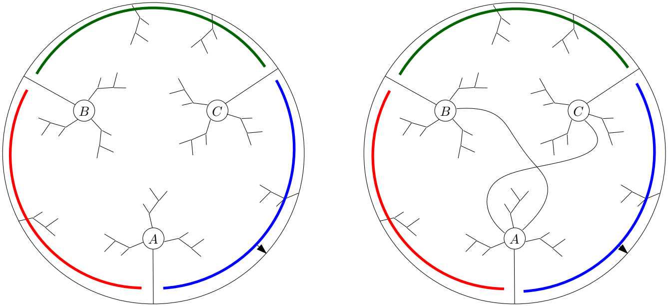
On Hurwitz maps, our asymptotic bijection consists in adding two edges;



To go from Hurwitz maps to permutation factorizations, we need to label vertices following the unique face of the map.

The relabeling permutation σ

On Hurwitz maps, our asymptotic bijection consists in adding two edges;



To go from Hurwitz maps to permutation factorizations, we need to label vertices following the unique face of the map.

Key observation. the contour order of the face changes: blue–red–green on the left and blue–green–red on the right (one can prove that pending trees are of size $\mathcal{O}_P(1)$). $\Rightarrow \sigma$ is close to $\widetilde{\sigma}$.

Thank you for your attention!