Formes limites de permutations aléatoires

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Introduction

Main topic: random permutations

- Classical questions: look at some statistics, like the number of cycles (of given length), longest increasing subsequences, ...
(usually for uniform or Ewens distributions)

- a more recent approach: look for a limit theorem for the renormalized "permutation matrix" (interesting for non-uniform or constrained models).

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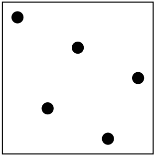
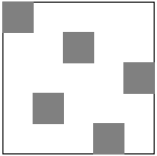
Here: we consider some constraints, called *pattern avoidance*.

Which notion of convergence? Permutons...

A permutation π can be represented by its diagram ($\sim$ permutation matrix) and mapped to a probability measure μ_π on $[0,1]^2$, called permuton.

$\pi = 52413 =$  $\mapsto$ $\mu_\pi =$ 

In μ_π, each small square has weight $1/n$ (i.e. density n).

We have a natural notion of limit for such objects: the weak convergence of measure.

Permutation patterns

Definition

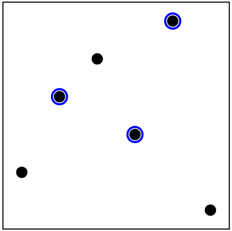
An occurrence of a pattern τ in σ is a subsequence $\sigma_{i_1} \dots \sigma_{i_k}$ that is order-isomorphic to τ, *i.e.* $\sigma_{i_s} < \sigma_{i_t} \Leftrightarrow \tau_s < \tau_t$.

Example (occurrences of 213)

2 4 5 3 6 1

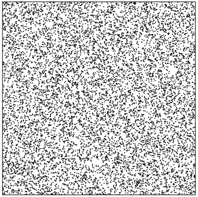
8 2 3 4 6 1 7 5

Visual interpretation

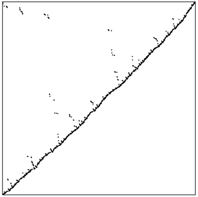


Pattern avoidance is a well-studied concept in enumerative combinatorics!

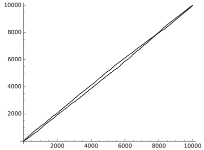
Uniform random permutations avoiding some patterns



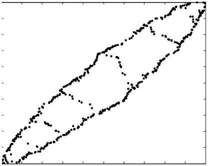
no constraints



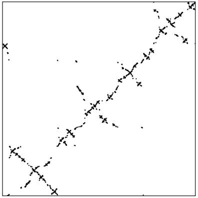
Av(231) (©MM)



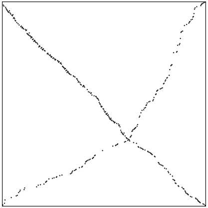
Av(321) (©HRS)



Av(4231) (©NM)



Av(2413,3142) (©MM)



Av(. . .) (©MM)

MM: Mickael Maazoun, HRS:Hoffman–Rizollo–Slivken, NM:Neil Madras

Detour: an operad structure on permutations

Well-known: the set S_n of permutations of size n is a group for the composition operation.

Less known: the set $\biguplus_{n \geq 1} S_n$ of permutations of all sizes is an operad for the substitution operation.

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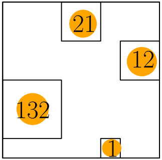
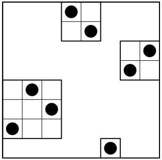
Definition (substitution)

Let θ be a permutation of size d and $\pi^{(1)}, \dots, \pi^{(d)}$ be permutations. The diagram of the permutation $\theta[\pi^{(1)}, \dots, \pi^{(d)}]$ is obtained by replacing the i-th dot in the diagram of θ with the diagram of $\pi^{(i)}$ (for each i).

$2413[132, 21, 1, 12] =$ 

$=$ 
$= 24387156$

Simple permutations and substitution decomposition

Definition

A permutation is called *simple* if it cannot be obtained as a nontrivial substitution.

Examples: 12, 21, 3142, 2413, 25314, , ...

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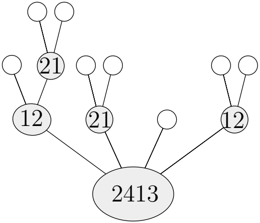
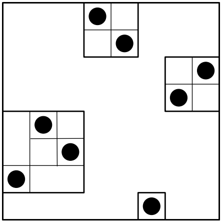
It's an analogue notion to that of prime numbers. In both cases, there are "factorization theorems":

- an integer can be uniquely represented as a multiset of prime numbers;
- a permutation can be (almost) uniquely represented as a "tree of permutations" (we call this its substitution decomposition)

We get trees and not multisets since we have an operad structure, and not a commutative monoid (as for integers).

Substitution decomposition and separable permutations





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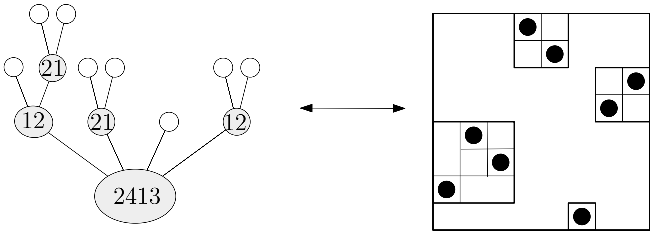
Inner nodes of the decomposition tree are labelled with simple permutations.

Substitution decomposition and separable permutations



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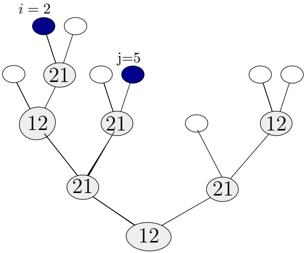
Inner nodes of the decomposition tree are labelled with simple permutations.

Proposition

$Av(2413, 3142)$ is the set of permutations whose decomposition trees contain only nodes labelled with 12 and 21.

These are called separable permutations.

Quizz



Problem

Given the tree T associated with a separable permutation σ and integers $i < j$, how to determine whether $\sigma(i) < \sigma(j)$?

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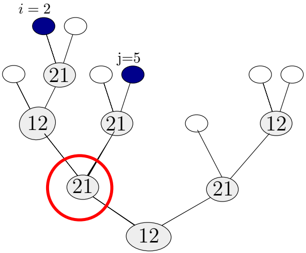

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Answer: look at the decoration of the first common ancestor between the i-th leaf and the j-th leaf.
In the example, it is 21 so $\sigma(2) > \sigma(5)$.

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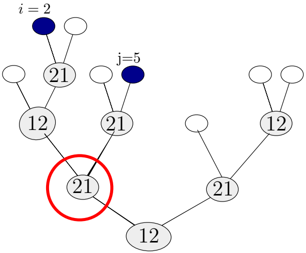

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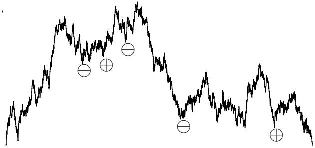
Write $i <_T j$ (resp. $i >_T j$) when $i < j$ and their common ancestor is 12 (resp. 21). We can reconstruct σ from this order:

$$\sigma(i) = 1 + \left|\{j : j <_T i\}\right|$$

The limiting object: the Brownian separable permuton



(e, S)

1. e is a Brownian excursion and $S : \mathrm{LocalMin}(e) \to \{\oplus, \ominus\}$ is an assignment of i.i.d. random signs to local minima of e (the probability to get $\oplus$ is $p \in (0,1)$).
   (the Brownian excursion encodes the limit of the trees, its local minima corresponding to branching points in the trees)

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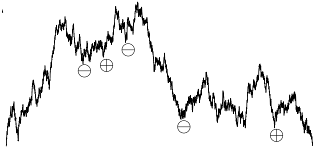
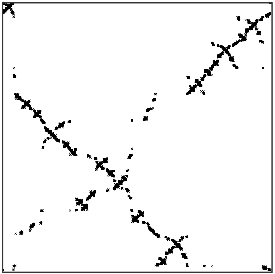


(e, S)

$\mapsto <_{(e,S)} \mapsto \tau \mapsto$



$\boldsymbol{\mu}_p = (x, \tau(x))_\star(\mathrm{Leb}([0,1]))$

1. If $x < y$ in $[0,1]$, we set $x <_{(e,S)} y$ if $S(\mathrm{argmin}_{[x,y]}\, e) = \oplus$; and $y <_{(e,S)} x$ if $S(\mathrm{argmin}_{[x,y]}\, e) = \oplus$.
2. We define a function $\tau : [0,1] \to [0,1]$ by $\tau(x) = \mathrm{Leb}\big(\{y : y <_{(e,S)} x\}\big)$.
3. The Brownian separable permuton $\boldsymbol{\mu}_p$ is the corresponding permuton.

Limits of separable permutations

Theorem (Bassino-Bouvel-F.-Gerin-Pierrot, 2018)

For each n, let $\boldsymbol{\sigma}_n$ be a uniform random separable permutation of size n. Then $\mu_{\boldsymbol{\sigma}_n}$ converges in distribution to $\boldsymbol{\mu}_{1/2}$.

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Similar results for other classes $\mathrm{Av}(B)$ (based on the algebraic properties of the class w.r.t. the operad structure):

- uniform random permutations $\boldsymbol{\sigma}_n$ in substitution-closed classes $\mathrm{Av}(B)$ tend to $\boldsymbol{\mu}_p$ (under some analytic conditions; p depends on he class).
- If $\mathrm{Av}(B)$ is finitely generated (i.e. contains finitely many simple), there is a dichotomy (see next slide; again with some technical conditions).

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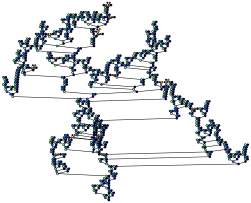
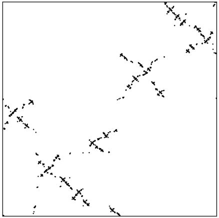
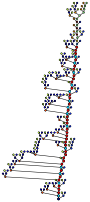
The dichotomy for finitely generated classes

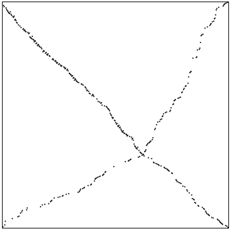
"Essentially branching case"





"Essentially linear case"





Any questions? I have one...

Let σ be a permutation of size n. Do there exist polynomials P_1, P_2, ..., P_n such that

- $P_1(0) = \cdots = P_n(0) = 0$;
- for small $x < 0$, we have
$$P_1(x) < P_2(x) < \cdots < P_n(x);$$
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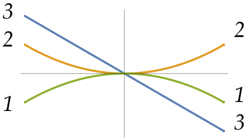
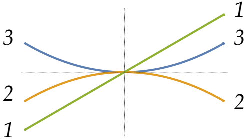
$$P_{\sigma(1)}(x) < P_{\sigma(2)}(x) < \cdots < P_{\sigma(n)}(x)?$$





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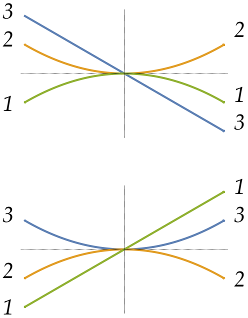

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Answers in office 221 (Nancy). Thank you for your attention.