Large permutations and permutons

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Introduction

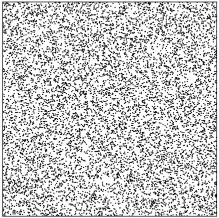
Main topic: random permutations

- Classical questions: look at some statistics, like the number of cycles (of given length), pattern occurrences, longest increasing subsequences, ...
  (usually for uniform, Ewens or Mallows distributions)

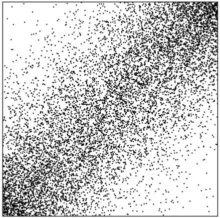
- a more recent approach: look for a limit for the rescaled permutation matrix; such limits are called permutons.
  (interesting for non-uniform models or constrained permutations)

This talk: *very biased* presentation of the notion of permutons and some literature on them.

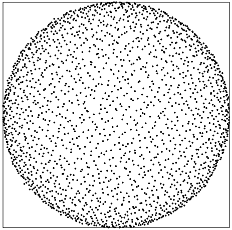
A few random permutations



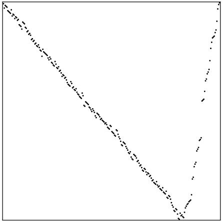
Uniform

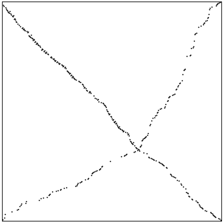


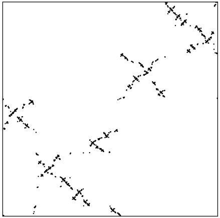
Mallows ($\mathbb{P}(\sigma) \propto q^{\mathrm{inv}(\sigma)}$)



Sorting network,
half way (©AHRV '07)







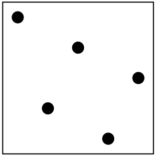
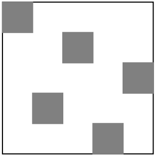
Uniform random pattern-avoiding permutations

The theory of permutons

(Hoppen, Kohayakawa, Moreira, Rath, Sampaio, '13)

How to look at large permutations?

A permutation π can be encoded as a probability measure μ_π on $[0,1]^2$.

$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

In μ_π, each small square has weight $1/n$ (i.e. density n).

We have a natural notion of limit for such objects: the weak convergence.
This defines a nice compact Polish space.

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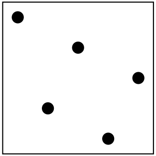
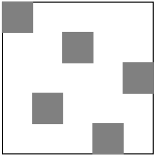
$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

In μ_π, each small square has weight $1/n$ (i.e. density n).

Note: the projection on μ_π on each axis is the Lebesgue measure on $[0,1]$ (in other words, μ_π has uniform marginals).
$\rightarrow$ potential limits also have uniform marginals.

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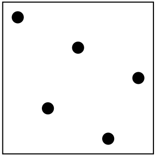
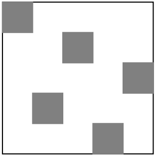
$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

In μ_π, each small square has weight $1/n$ (i.e. density n).

Definition

A permuton is a probability measure on $[0,1]^2$ with uniform marginals.

Next few slides: connection with permutation patterns.

Permutation patterns

Definition

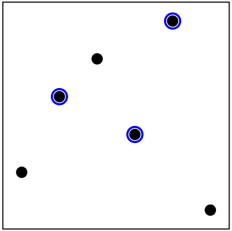
An occurrence of a pattern τ in σ is a subsequence $\sigma_{i_1} \dots \sigma_{i_k}$ that is order-isomorphic to τ, *i.e.* $\sigma_{i_s} < \sigma_{i_t} \iff \tau_s < \tau_t$.

Example (occurrences of 2 1 3)

2 4 5 3 6 1

8 2 3 4 6 1 7 5

Visual interpretation



Pattern density in permutations and permutons

If τ and σ are permutations of size k and n, resp., we set

$$\widetilde{\mathrm{occ}}(\tau,\sigma) := \binom{n}{k}^{-1} \cdot \# \left\{ \begin{array}{c} \text{occurrences of} \\ \tau \text{ in } \sigma \end{array} \right\} \in [0,1].$$

In other terms: take k elements uniformly at random in σ, the probability to find a pattern τ is $\widetilde{\mathrm{occ}}(\tau,\sigma)$.

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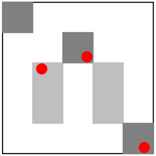
This probabilistic interpretation extends to permutons: replacing σ with a permuton μ

$$\widetilde{\text{occ}}(\tau, \mu) := \mathbb{P}^{\mu}(U^{(1)}, \cdots, U^{(k)} \text{ form a pattern } \tau),$$

where $U^{(1)}, \cdots, U^{(k)}$ are i.i.d. points in $[0,1]^2$ with distribution μ.



a “231 pattern” in a permuton

Pattern density convergence and permuton convergence

Theorem (Hoppen, Kohayakawa, Moreira, Rath, Sampaio, 2013)

For each $n \geq 1$, let σ_n be a permutation of size n. TFAE

(a) μ_{σ_n} *converges to some permuton* μ.

(b) *For every pattern* π, *the proportion* $\widetilde{\mathrm{occ}}(\pi, \sigma_n)$ *tends to some* δ_π

Moreover, if both hold, $\delta_\pi = \widetilde{\mathrm{occ}}(\pi, \mu)$.

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Theorem (Bassino-Bouvel-F.-Gerin-Maazoun-Pierrot, 17)

For each $n \geq 1$, let $\boldsymbol{\sigma}_n$ be a random permutation of size n. TFAE

(a) $\mu_{\boldsymbol{\sigma}_n}$ *converges in distribution to some random permuton* $\boldsymbol{\mu}$ *(warning: random measures around!)*

(b) *For every pattern* π, *there is a* $\Delta_\pi \geq 0$ *such that*

$$\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)] \xrightarrow{n\to\infty} \Delta_\pi.$$

Moreover, if both hold, $\Delta_\pi = \mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\mu})]$.

Some convergent models of random permutations

(and nice pictures)

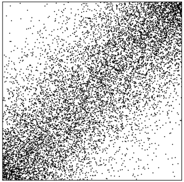
Mallows permutations

Mallows model on S_n: $\mathbb{P}(\sigma_n) \propto q_n^{\mathrm{inv}(\sigma_n)}$,
where $\mathrm{inv}(\sigma) = \#\{(i,j) \text{ with } i < j \text{ and } \sigma(i) > \sigma(j)\}$.

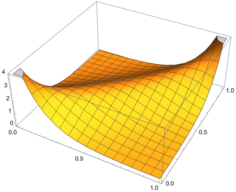
Theorem (Starr, '09)

Take $q_n = 1 - \beta/n$. Then $\mu_{\boldsymbol{\sigma}^{(n)}}$ converge to the deterministic permuton with density

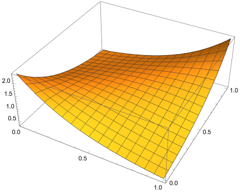
$$u(x,y) = \frac{(\beta/2)\sinh(\beta/2)}{\left(e^{\beta/4}\cosh(\beta[x-y]/2) - e^{-\beta/4}\cosh(\beta[x+y-1]/2)\right)^2}.$$



Simulation ($n = 10000$, $\beta = 6$)



$\beta = 6$



$\beta = 2$

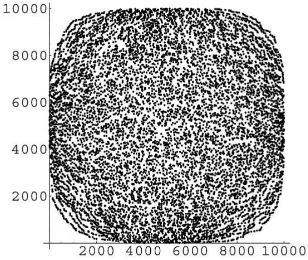
Erdős-Szekeres extremal permutations

An Erdős-Szekeres extremal permutation is a permutation of size n^2 that has no monotone subsequence of size $n+1$.

Theorem (Romik, '06)

Let $\boldsymbol{\sigma}_n$ be a uniform random Erdős-Szekeres extremal permutation of size n^2. Then $\boldsymbol{\sigma}_n$ converges to a deterministic permuton supported by

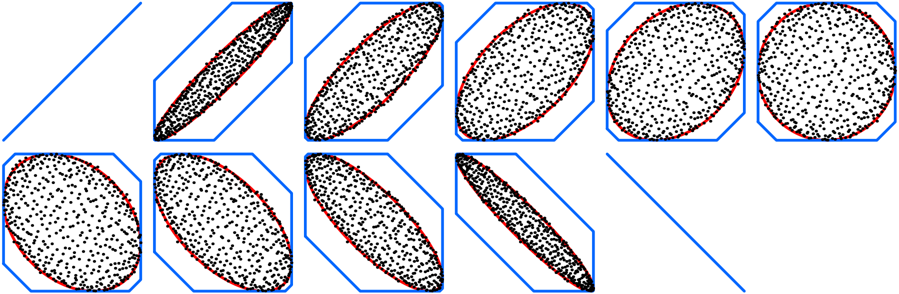
$$\left\{x, y \in [0,1]^2 : (x^2 - y^2)^2 + 2(x^2 + y^2) \leq 3\right\}$$



© Romik

Random sorting networks

A sorting network is a minimal path going from the identity permutation to the reverse permutation, switching two adjacent entries at each step.



Random sorting network, ©Angel, Holroyd, Romik and Virag ('07)

A formula for the limiting process in the space of permutons was conjectured by Angel, Holroyd, Romik and Virag ('07) and proved by Dauvergne ('18).

And more...

- Random permutations in grid classes (Bevan '15), Square permutations (Borga, Slivken '19), various exponentially biased models (Mukherjee '16, Bouvel/Nicaud/Pivoteau '19), ...
- Large deviation principle for uniform random permutations in the space of permutons (Trashorras, '08, Kenyon, Král, Radin, Winkler, '15).
- Asymptotics of the number of cycles of fixed length (Mukherjee, '16), of the length of the longest increasing subsequence (Mueller, Starr, '13) and of the total displacement (Bevan, Winkler, '19) in Mallows permutations using the permuton limit.

Limits of permutation classes with a finite specification

(joint work with Bouvel, Bassino, Gerin, Maazoun, Pierrot)

Permutation classes

Definition

A set $\mathcal{C}$ of permutations (of all sizes) is a class if for all permutations π in $\mathcal{C}$, and all *patterns* τ of π, τ is also in $\mathcal{C}$.

Equivalently, a class is the set of permutations avoiding given patterns.

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- Traditionally analyzed from an enumerative point of view: how many permutations of size n are there in a given class?
- More recently from a probabilistic point of view: what does a uniform random permutation in a given class look like? (Atapour, Bevan, Borga, Dokos, Hoffman, Janson, Liu, Madras, Miner, Pak, Pehlivan, Rizzolo, Slivken, Stufler, Yldrm, . . .)

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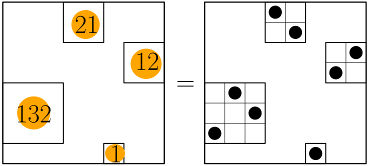
Substitution in permutations (1/2)

Definition of substitution

Let θ be a permutation of size d and $\pi^{(1)}, \dots, \pi^{(d)}$ be permutations. The diagram of the permutation $\theta[\pi^{(1)}, \dots, \pi^{(d)}]$ is obtained by replacing the i-th dot in the diagram of θ with the diagram of $\pi^{(i)}$ (for each i).

$2413[132, 21, 1, 12] =$



$= 24387156$

Definition

A permutation is called simple if it cannot be obtained as a nontrivial substitution.

Examples: 12, 21, 3142, 2413, 25314, …

Substitution in permutations (2/2)

Proposition (Albert, Atkinson, '05)

Every permutation σ of size $n \geq 2$ can be uniquely decomposed as either:

- $\alpha[\pi^{(1)}, \dots, \pi^{(d)}]$, *where α is simple of size $d \geq 4$,*
- $12[\pi^{(1)}, \pi^{(2)}]$, *where $\pi^{(1)}$ is 12-indecomposable,*
- $21[\pi^{(1)}, \pi^{(2)}]$, *where $\pi^{(1)}$ is 21-indecomposable.*

Not very interesting for uniform random permutation: the simple permutation α has typically size $n - O(1)$.

But interesting for permutations in classes! It has been used for enumerating many classes.

Classes with finitely many simple permutations (1/2)

Assume we have a finite number of simple permutations in a class $\mathcal{C}$.

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First thought: great, the substitution decomposition gives us a system of equation for the class

$$\begin{cases} \mathcal{C} & \stackrel{?}{=} \{\bullet\} \uplus 12[\mathcal{C}^{\mathrm{not}\oplus}, \mathcal{C}] \uplus 21[\mathcal{C}^{\mathrm{not}\ominus}, \mathcal{C}] \uplus \left(\biguplus_{|\alpha| \geq 4} \alpha[\mathcal{C}, \dots, \mathcal{C}] \right) \\ \mathcal{C}^{\mathrm{not}\oplus} & \stackrel{?}{=} \{\bullet\} \uplus 21[\mathcal{C}^{\mathrm{not}\ominus}, \mathcal{C}] \uplus \left(\biguplus_{|\alpha| \geq 4} \alpha[\mathcal{C}, \dots, \mathcal{C}] \right) \\ \mathcal{C}^{\mathrm{not}\ominus} & \stackrel{?}{=} \{\bullet\} \uplus 12[\mathcal{C}^{\mathrm{not}\oplus}, \mathcal{C}] \uplus \left(\biguplus_{|\alpha| \geq 4} \alpha[\mathcal{C}, \dots, \mathcal{C}] \right). \end{cases}$$

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☹ not quite, we can create forbidden patterns in the substitution!

$\rightarrow$ we need to replace some of the $\mathcal{C}$ above by some subfamilies of $\mathcal{C}$, consider cases, resolve ambiguities and iterate...

Classes with finitely many simple permutations (2/2)

Theorem (Bassino-Bouvel-Pierrot-Pivoteau-Rossin '17)

Any class $\mathcal{C}$ with finitely many simple permutations admits a finite combinatorial specification of the form

$$\mathcal{C}_i \;=\; \varepsilon_i\{\bullet\} \;\uplus\; \biguplus_{\alpha\in\mathcal{S}_{\mathcal{C}_i}} \;\biguplus_{(k_1,\dots,k_{|\alpha|})\in K^i_\alpha} \alpha[\mathcal{C}_{k_1},\cdots,\mathcal{C}_{k_{|\alpha|}}] \qquad (0\leq i\leq d) \quad (1)$$

where the $\mathcal{C}=\mathcal{C}_0\supset\mathcal{C}_1,\cdots,\mathcal{C}_d$ and the ε_i are in $\{0,1\}$.

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where the $\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1, \cdots, \mathcal{C}_d$ and the ε_i are in $\{0, 1\}$.

The system can be obtained algorithmically (implemented by Maazoun).

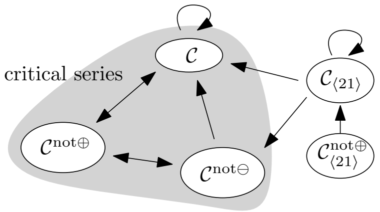
$\rightarrow$ gives an algebraic system of equations for the GF of $\mathcal{C}$.

$\rightarrow$ yields a random sampler for the class $\mathcal{C}$ (used for simulations in the introduction).

Finite specification: the example of Av(132)

$$\begin{cases} \mathcal{C} & = \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\text{not}\oplus}, \mathcal{C}_{\langle 21 \rangle}] \quad \uplus \quad \ominus[\mathcal{C}^{\text{not}\ominus}, \mathcal{C}] \\ \mathcal{C}^{\text{not}\oplus} & = \{\bullet\} \quad \uplus \quad \ominus[\mathcal{C}^{\text{not}\ominus}, \mathcal{C}] \\ \mathcal{C}^{\text{not}\ominus} & = \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\text{not}\oplus}, \mathcal{C}_{\langle 21 \rangle}] \\ \mathcal{C}_{\langle 21 \rangle} & = \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\text{not}\oplus}_{\langle 21 \rangle}, \mathcal{C}_{\langle 21 \rangle}] \\ \mathcal{C}^{\text{not}\oplus}_{\langle 21 \rangle} & = \{\bullet\}. \end{cases}$$

Associated dependency graph indicating families with maximal growth rate (called critical families):



Main theorem

Theorem (BBFGMP, '19)

Let $\mathcal{C}$ be a family of permutations with a finite analytic specification (e.g. a permutation class with finitely many simple permutations). Assume that the dependency graph restricted to critical families is strongly connected (plus some weak aperiodicity assumption).

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essentially linear case *If the specification contains no products of critical families, then a uniform random permutation in the class converges to an X-permuton with computable parameters.*

essentially branching case *If the specification contains a product of critical families, then a uniform random permutation in the class converges to a Brownian separable permuton with computable parameters.*

Description of the limit permutons and examples in the next few slides...

Is Av(231) essentially linear or branching?

$$\begin{cases} \mathcal{C} & = \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\mathrm{not}\oplus}, \mathcal{C}_{\langle 21 \rangle}] \quad \uplus \quad \ominus[\mathcal{C}^{\mathrm{not}\ominus}, \mathcal{C}] \\ \mathcal{C}^{\mathrm{not}\oplus} & = \{\bullet\} \quad \uplus \quad \ominus[\mathcal{C}^{\mathrm{not}\ominus}, \mathcal{C}] \\ \mathcal{C}^{\mathrm{not}\ominus} & = \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\mathrm{not}\oplus}, \mathcal{C}_{\langle 21 \rangle}] \\ \mathcal{C}_{\langle 21 \rangle} & = \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\mathrm{not}\oplus}_{\langle 21 \rangle}, \mathcal{C}_{\langle 21 \rangle}] \\ \mathcal{C}^{\mathrm{not}\oplus}_{\langle 21 \rangle} & = \{\bullet\}. \end{cases}$$

Critical series: $\mathcal{C}$, $\mathcal{C}^{\mathrm{not}\oplus}$, $\mathcal{C}^{\mathrm{not}\ominus}$.

The specification contains a product of critical classes

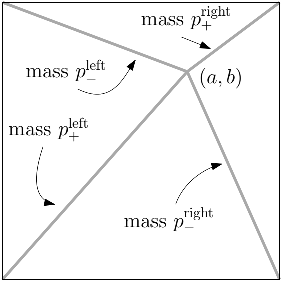
$\longrightarrow$ essentially branching case.

The X-permuton

Parameter: a quadruple of sum 1

$$(p_+^{\text{left}}, p_+^{\text{right}}, p_-^{\text{left}}, p_-^{\text{right}}).$$

We set $a = p_+^{\text{left}} + p_-^{\text{left}}$
and $\quad b = p_+^{\text{left}} + p_-^{\text{right}}$
(to ensure the uniform marginal condition).



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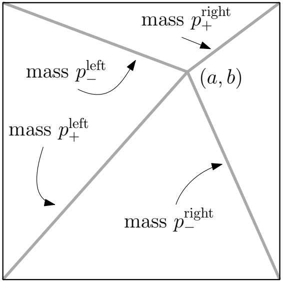
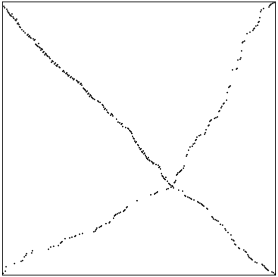
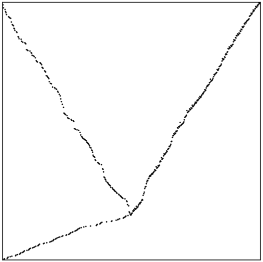


Note: this is a deterministic permuton.

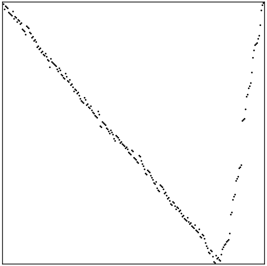
The essentially linear case: examples



$\mathrm{Av}(2413, 3142, 2143, 34512)$



$\mathrm{Av}(231, 21543)$

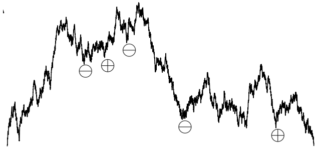


$\mathrm{Av}(2413, 1243, 2341, 41352, 531642)$

Note: in the second (resp. third) case, one (resp. two consecutive) parameters are 0. Diagonals are also degenerate X-permutons (with 2 opposite or 3 parameters equal to 0).

The Brownian separable permuton (Maazoun '17)

Parameter: $p \in [0, 1]$

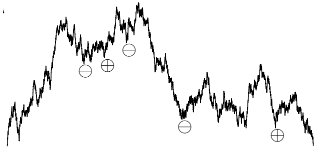


(e, S)

- e is a Brownian excursion and S : $\mathrm{LocalMin}(e) \to \{\oplus, \ominus\}$ is a independent assignment of signs to local minima of e (the probability to get a $\oplus$ is p).

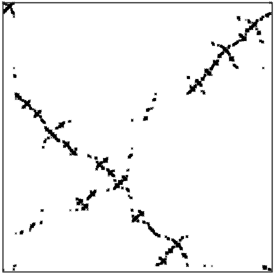
The Brownian separable permuton (Maazoun '17)

Parameter: $p \in [0,1]$



(e, S)

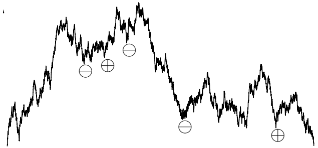
$\mapsto \quad \sigma \quad \mapsto$



$\boldsymbol{\mu} = (x, \sigma(x))_\star(\mathrm{Leb}([0,1]))$

- $\sigma : [0,1] \to [0,1]$ is the unique Lebesgue preserving function s.t. (x, y) is an inversion if and only if the sign of $\min_{[x,y]} e$ is $\ominus$.
- The Brownian separable permuton is the "graph of the function σ".

Parameter: $p \in [0,1]$



(e, S)

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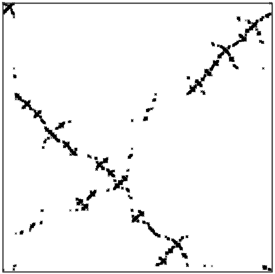
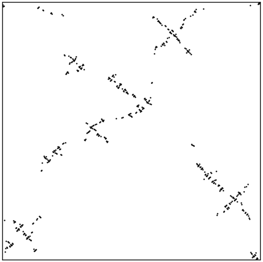


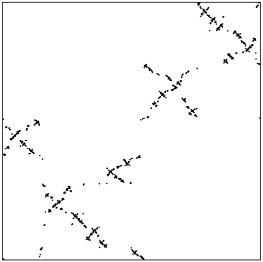
$\boldsymbol{\mu} = (x, \sigma(x))_{\star}(\mathrm{Leb}([0,1]))$

Note: this a random permuton. No concentration phenomenon here.

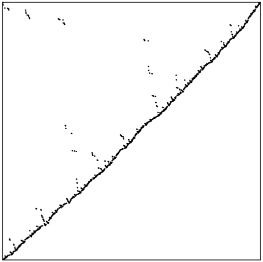
The essentially branching case: examples



$\mathrm{Av}(2413, 3142)$
separable permutations



$\mathrm{Av}(2413, 31452,$
$41253, 41352, 531246)$



$\mathrm{Av}(231)$

The limit in the last case is a degenerate Brownian permuton with $p = 1$, that is the diagonal of the square. This convergence to the diagonal (and much more precise results) was already known.

A word on the proofs

1. Reminder: enough to prove that, for any τ,

$$\mathbb{E}\big[\,\widetilde{\mathrm{occ}}(\tau, \boldsymbol{\sigma}_n)\big] \to \mathbb{E}\big[\,\widetilde{\mathrm{occ}}(\tau, \boldsymbol{\nu})\big],$$

where $\boldsymbol{\nu}$ is the targeted limit random permuton.

A word on the proofs

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   where $\boldsymbol{\nu}$ is the targeted limit random permuton.

2. The RHS can be evaluated easily (elementary for X-permuton, using some results on Brownian excursion for the Brownian one).

3. The LHS can be computed combinatorially:

$$\mathbb{E}[\widetilde{\mathrm{occ}}(\pi,\boldsymbol{\sigma}_n)] = \frac{\#\{\sigma \in \mathcal{C}_n, I \subset [n] : \mathrm{pat}_I(\sigma) = \pi\}}{\binom{n}{k}\ |\mathcal{C}_n|}.$$

   We will estimate that through analytic combinatorics (see Benedikt's talk for a more probabilistic approach).

Analytic combinatorics

The strongly connectedness hypothesis ensures that

- in the essentially linear case,

$$C(z) \sim a \frac{1}{1-\frac{z}{\rho}}, \text{ implying } |\mathcal{C}_n| \sim a\rho^{-n}.$$

- in the branching case,

$$C(z) \sim a - b\sqrt{1-\tfrac{z}{\rho}}, \text{ implying } |\mathcal{C}_n| \sim \frac{b}{2\sqrt{\pi}} n^{3/2} \rho^{-n}$$

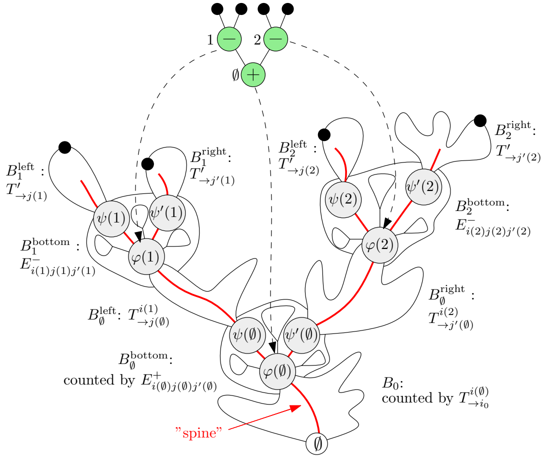
The difficulty is to estimate

$$\{\#\{\sigma \in \mathcal{C}_n, I \subset [n] : \mathrm{pat}_I(\sigma) = \pi\}\}.$$

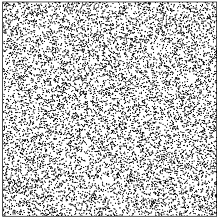
We need to write some equations for the corresponding generating function and to find the behavior at the singularity.

A picture of a combinatorial decomposition

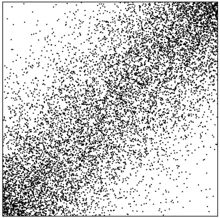
(where permutations are encoded by trees thanks to the specification.)



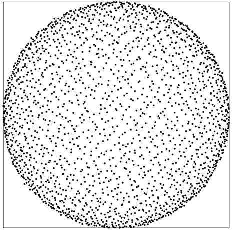
Thank you for your attention



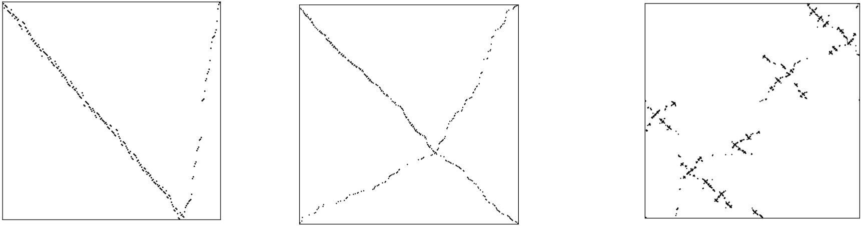
Uniform



Mallows ($\mathbb{P}(\sigma) \propto q^{\mathrm{inv}(\sigma)}$)



Sorting network,
half way (©AHRV '07)

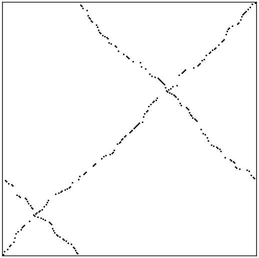


Uniform random pattern-avoiding permutations

Extra slide 1: is the strong connectivity condition necessary?

Yes!

Here is a class with no simple permutations and a “double X” limit:



Av(2413, 3142, 3412, 214365, 52143, 32541)

We can treat such examples on a case-by-case basis from their finite specification, but we have no general theorem!

Extra slide 2: the *intensity* of the Brownian permuton

Since the Brownian permuton $\boldsymbol{\mu}_p$ is a random measure, we can consider its intensity measure $\mathbb{E}\boldsymbol{\mu}_p$, defined by

$$(\mathbb{E}\boldsymbol{\mu}_p)(R) = \mathbb{E}(\boldsymbol{\mu}(R)), \text{ for any rectangle } R \subseteq [0,1]^2.$$

Theorem (Maazoun '17)

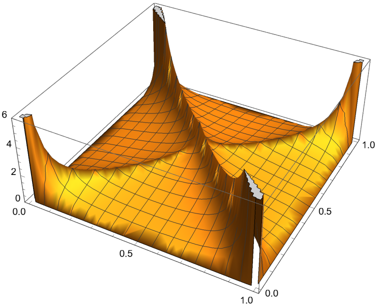
The intensity measure $\mathbb{E}\boldsymbol{\mu}_p$ has density w.r.t to Lebesgue measure

$$f_p(x,y) = \int_{\max(0,x+y-1)}^{\min(x,y)} \frac{3p^2(1-p)^2 da}{2\pi(a(x-a)(1-x-y+a)(y-a))^{3/2}\left(\frac{p^2}{a}+\frac{(1-p)^2}{(x-a)}+\frac{p^2}{(1-x-y+a)}+\frac{(1-p)^2}{(y-a)}\right)^{5/2}}.$$

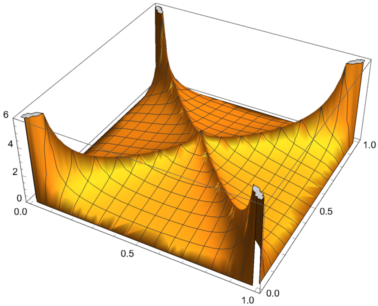
Concretely, if $\boldsymbol{\sigma}_n$ tends to $\boldsymbol{\mu}_p$, then, for any rectangle $R \subseteq [0,1]^2$

$$\mathbb{E}\big[\#\{(i,j) \in nR : \sigma(i) = j\}\big] \sim n \int_{(x,y)\in R} f_p(x,y)dxdy.$$

Extra slide 2bis: picture of $\mathbb{E}\boldsymbol{\mu}_p$



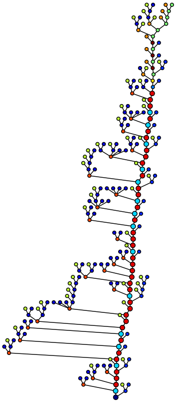
density of $\mathbb{E}\boldsymbol{\mu}_{.4}$



density of $\mathbb{E}\boldsymbol{\mu}_{.5}$

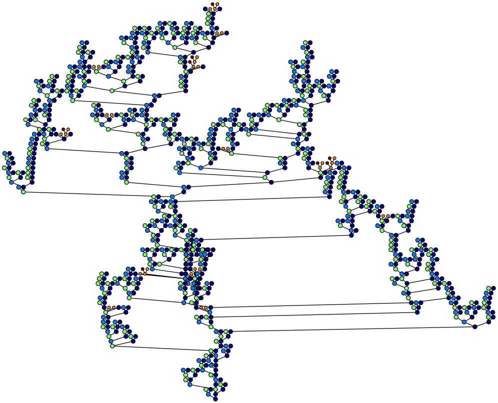
For $p = .5$, this function was found (under a different form) by Pak and Dokos, in the context of doubly alternating Baxter permutations.

Extra slide 3: underlying random trees



essentially linear case

Av(2413, 1243, 2341, 41352, 531642)



essential branching case

Av(2413, 31452, 41253, 41352, 531246)