Random factorizations of a long cycle into transpositions

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Our model

We consider minimal factorizations of a full cycle into transpositions.

$$\mathfrak{M}_n := \{(\tau_1, \dots, \tau_{n-1}) \text{ transpositions s.t. } \tau_1\tau_2\cdots\tau_{n-1} = (1, 2, \dots, n)\}$$

Well-known: $|\mathfrak{M}_n| = n^{n-2}$ (bijections with Cayley trees, parking functions)

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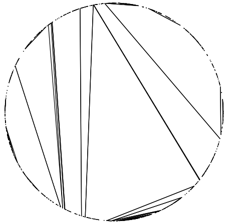
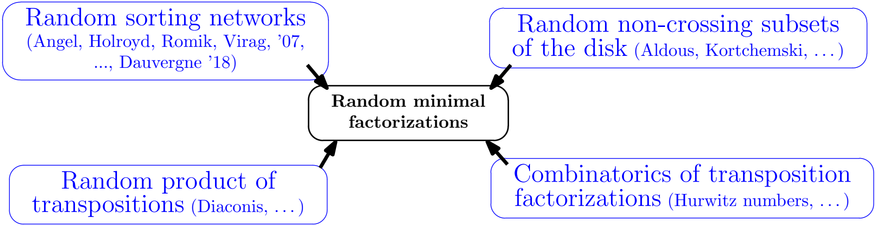
We are interested in partial products $\boldsymbol{\tau_1} \cdots \boldsymbol{\tau_k}$. They can be geometrically represented by non-crossing sets of chords.



Goal: find the limit of these random non-crossing objects.

Motivations



Geometric representations as non-crossing objects

With a minimal factorization $(\tau_1, \dots, \tau_{n-1})$ of size n and an integer $k \leq n-1$, we associate two non-crossing objects.

Example: take $n = 12$, $k = 6$, and

$\big((1,3),(6,12),(1,5),(7,12),(9,10),(11,12),(2,3),(4,5),(1,6),(8,11),(9,11)\big) \in \mathfrak{M}_{12}$

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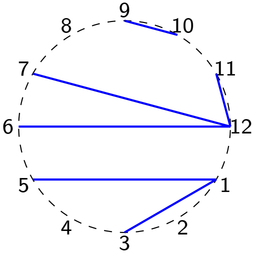
First construction $\boldsymbol{F_k^n}$

Each transposition corresponds to an edge.



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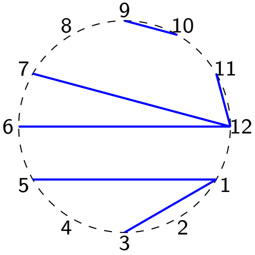
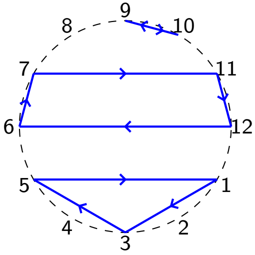


Second construction P_k^n

Draw the cycles of the partial product.



Main result

Theorem (F., Kortchemski, '18)

There exists a family of random compact subset $(\mathbf{L}_c)_{c\in[0,\infty]}$ of the unit disk such that the following holds. (We use Hausdorff metric.)

(i) *Assume that $K_n \to \infty$ and $\frac{K_n}{\sqrt{n}} \to c$, with $c < \infty$. Then*

$$\left(\boldsymbol{F}^{\boldsymbol{n}}_{\boldsymbol{K_n}}, \boldsymbol{P}^{\boldsymbol{n}}_{\boldsymbol{K_n}}\right) \xrightarrow[n\to\infty]{(d)} (\mathbf{L}_c, \mathbf{L}_c).$$

(ii) *Assume that $\frac{K_n}{\sqrt{n}} \to \infty$ and that $\frac{n-K_n}{\sqrt{n}} \to \infty$. Then*

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$$\boldsymbol{F}^{\boldsymbol{n}}_{\boldsymbol{K_n}} \xrightarrow[n\to\infty]{(d)} \mathbf{L}_\infty, \qquad \boldsymbol{P}^{\boldsymbol{n}}_{\boldsymbol{K_n}} \xrightarrow[n\to\infty]{(d)} \mathbf{L}_c.$$

What are the limit objects?

- $\mathbf{L}_0$ is the unit circle.
  i.e., according to our theorem, when $K_n = o(\sqrt{n})$, there is no macroscopic chord in the first K_n factors.

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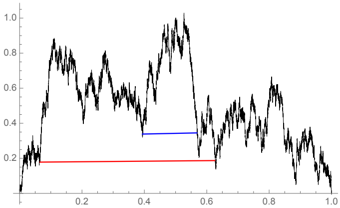
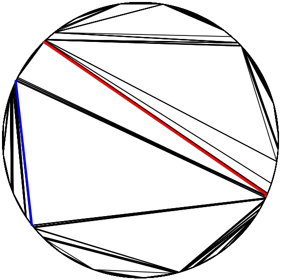
- $\mathbf{L}_0$ is the unit circle.
- $\mathbf{L}_\infty$ is Aldous' Brownian triangulation.
  Start from a Brownian excursion $\boldsymbol{X}_\infty^{\text{exc}}$ and draw a chord $[e^{-2\pi i s}, e^{-2\pi i t}]$ for each tunnel (s,t) in $\boldsymbol{X}_\infty^{\text{exc}}$.





Simulation of $\boldsymbol{X}_\infty^{\text{exc}}$ and $\boldsymbol{L}_\infty$.

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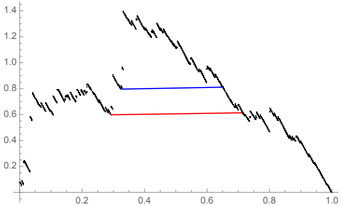

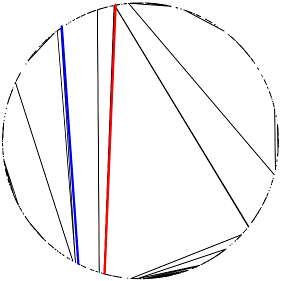

Simulation of $\boldsymbol{X}_5^{\text{exc}}$ and $\boldsymbol{L}_5$.

What are the limit objects?

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About the underlying Lévy process X_t:

$$\mathbb{E}e^{-\lambda X_t} = e^{t\,c^2\left(1-\sqrt{1+\frac{2\lambda}{c}}\right)+t\,\lambda c}, \qquad \lambda \geq 0.$$

It is spectrally positive, makes infinitely many jumps on all intervals, X_t = inverse Gaussian process + a negative drift $(-ct)$.

Illustrating movie

We take a uniform random minimal factorization of size $n = 20,000$.

The following movies displays the non-crossing forest $\boldsymbol{F^n_{K_n}}$ and the non-crossing partition $\boldsymbol{P^n_{K_n}}$ for $K_n = t(n-1)$ (t varying from 0 to 1);

Movie 1

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Movie 1

Our theorem describes the marginals of the above process.

☹ We have no limit result for the process itself.

A "concrete" corollary

For a transposition $\tau = (i,j)$ with $i < j$, set $w(\tau) := \min(j-i, n+i-j)$.

Corollary

Let $(\boldsymbol{\tau_1}, \dots, \boldsymbol{\tau_{n-1}})$ *be a uniform random factorization of size n. Then the random variable*

$$n^{-1} \max_{1 \le i \le K_n} w(\boldsymbol{\tau_i})$$

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(i) *tends to 0 in probability if $K_n = o(\sqrt{n})$;*

(ii) *tends to some non-trivial random variable ℓ_c if $\lim \frac{K_n}{\sqrt{n}} = c$ (for c in $(0, +\infty]$).*

The law of ℓ_∞ is explicit (computed by Aldous):

$$\frac{1}{\pi} \frac{3\theta - 1}{\theta^2 (1-\theta)^2 \sqrt{1 - 2\theta}} \mathbf{1}_{\frac{1}{3} \le \theta \le \frac{1}{2}} \, d\theta, \text{ where } \ell = 2\sin(\pi\theta).$$

Remarkable that the limit is the same for $K_n = \sqrt{n}\log(n)$ or $K_n = n - 1$!

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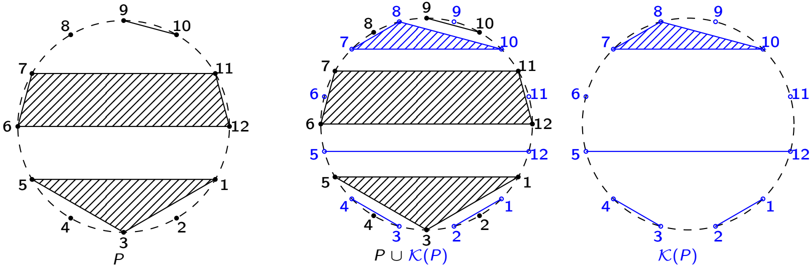
4. We analyse this random tree model.

An explicit formula for the distribution of $\mathbf{P^n_k}$

Given a non-crossing partition P, we can construct *its Kreweras complement* $\mathcal{K}(P)$. On an example,



Blocks of $\mathcal{K}(P)$ = white "faces" between blocks of P

Fact: blocks of P $\leftrightarrow$ cycles of σ.
blocks of $\mathcal{K}(P)$ $\leftrightarrow$ cycles of $\sigma^{-1}\,(1\ 2\ \ldots\ n)$

An explicit formula for the distribution of $\mathbf{P_k^n}$

Given a non-crossing partition P, we can construct *its Kreweras complement* $\mathcal{K}(P)$.

Proposition

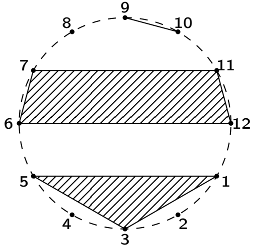
Fix $1 \leq k \leq n-1$ and let P be a non-crossing partition with $n-k$ blocks. Then

$$\mathbb{P}\left(\mathbf{P_k^n} = P\right) = \frac{k!(n-k-1)!}{n^{n-2}} \cdot \left(\prod_{B \in P \uplus \mathcal{K}(P)} \frac{|B|^{|B|-2}}{(|B|-1)!} \right).$$

The proof is easy; it relies on the fact that the number of minimal factorizations of any permutation into transpositions is explicit.

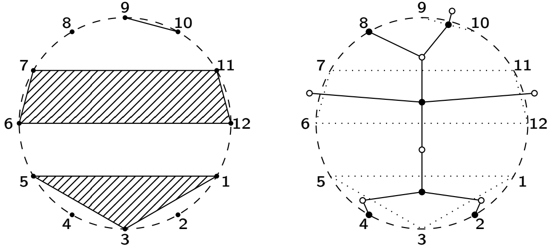
A bijection $\mathcal{T}$ from non-crossing partitions to trees

1. Start from a non-crossing partition;



A bijection $\mathcal{T}$ from non-crossing partitions to trees

1. Start from a non-crossing partition;
2. Associate a black vertex to each block, a white vertex to each block of the Kreweras complement, and join neighbour blocks;
   (We get a properly bicolored plane tree with labeled black corners.)



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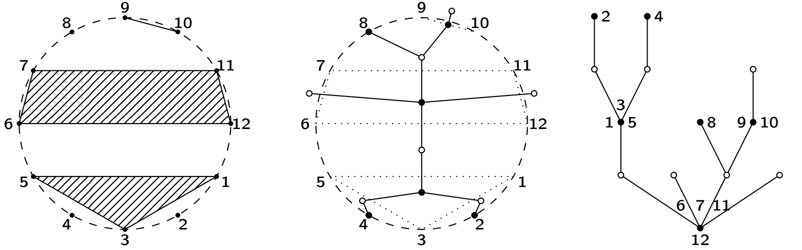

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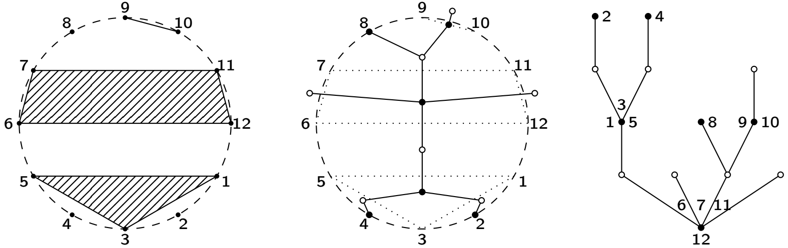

Inverse: label black corners by the order of visit of the contour of trees. Each block corresponds to the corners of some black vertex.

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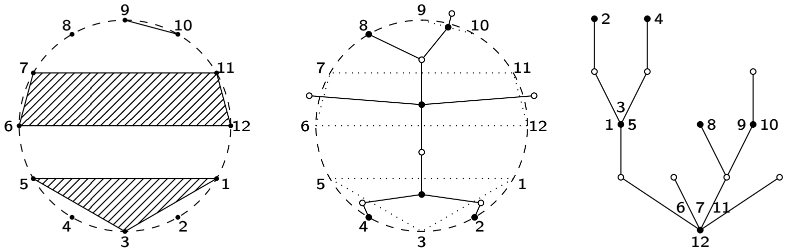

Here, and throughout this talk, black (resp. white) vertices are vertices at even (resp. odd) height.

Conditioned Galton-Watson trees

Reminder:
if P is a non-crossing partition of n with $n-k$ blocks,

$$\mathbb{P}(\mathbf{P_k^n} = P) \propto \prod_{B \in P \uplus \mathcal{K}(P)} \frac{|B|^{|B|-2}}{(|B|-1)!}.$$

Conditioned Galton-Watson trees

Using the bijection:
if T is a tree with $n-k$ black and $k+1$ white vertices, then

$$\mathbb{P}(\mathcal{T}(\mathbf{P_k^n}) = T) \propto \prod_{v \in V_T} \frac{(d_v+1)^{d_v-1}}{d_v!},$$

where d_v is the out-degree of v (small correction needed at the root).

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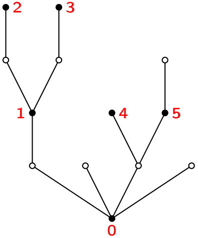
☹ When $k = \Theta(\sqrt{n})$, the conditioning event has exponentially small probability....

$\rightarrow$ we see T as a conditioned bi-type Galton-Watson tree, with offspring distributions depending on n.

Łukasiewicz paths



- Start with a tree T and label its black vertices in depth-first search order.

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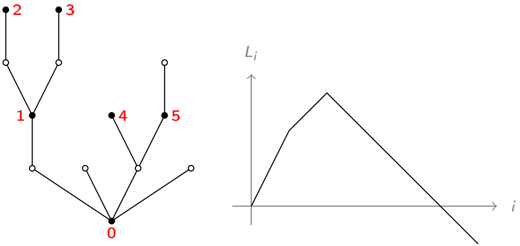

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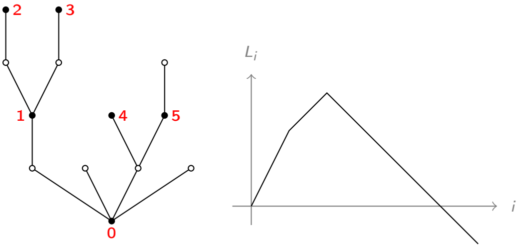

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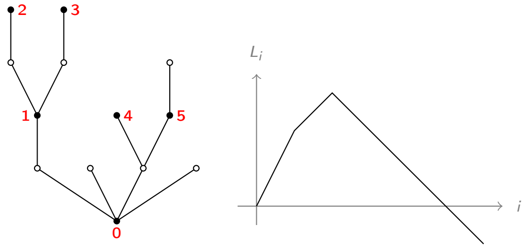
Why Łukasiewicz paths?

For many models of (bitype) conditioned GW trees, the Łukasiewicz path is a conditioned random walk, which we can analyse asymptotically.

Łukasiewicz paths



In our model, the Łukasiewicz path of $\mathcal{T}(\boldsymbol{P}^{\boldsymbol{n}}_{\boldsymbol{K_n}})$ tends (after normalization) to the excursion process $\boldsymbol{X}_c^{\text{exc}}$.

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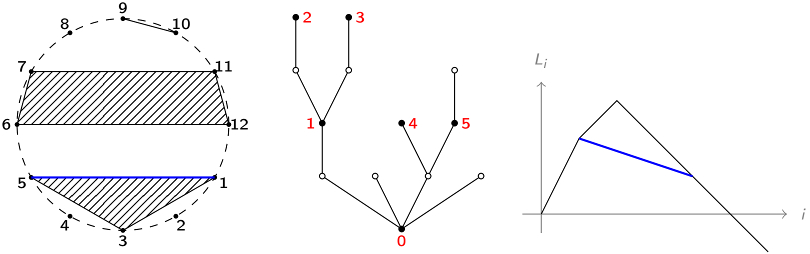

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Moreover, chords in P **asymptotically** correspond to tunnels in the Łukasiewicz path of $\mathcal{T}(P)$.

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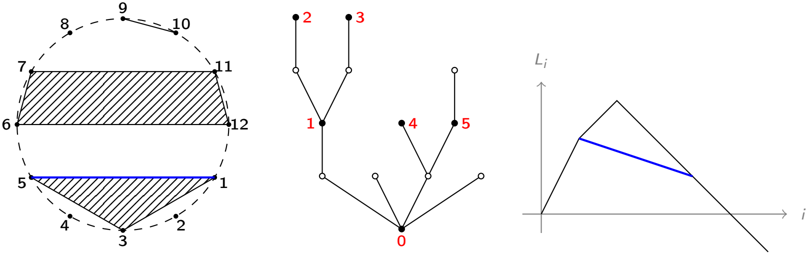

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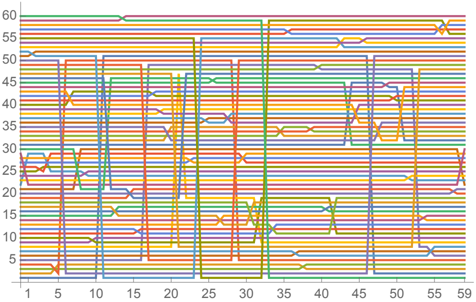
Moreover, chords in P **asymptotically** correspond to tunnels in the Łukasiewicz path of $\mathcal{T}(P)$.

Conclusion: $\boldsymbol{P}^{\boldsymbol{n}}_{\boldsymbol{K_n}}$ tends to $\boldsymbol{L}_c$.

Local convergence of trajectories (1/3)

Our "scaling limit" result gives no information on what happens for some fixed elements, *e.g.* what is the trajectory of a given i? The number of transpositions containing it?

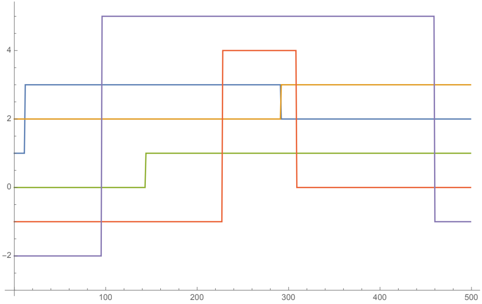


Local convergence of trajectories (2/3)

We work with factorizations of the cycle

$$(-\lfloor (n-1)/2 \rfloor, \dots, 0, 1, \dots, \lfloor n/2 \rfloor)$$

and look around $i = 0$:



Trajectories of all i in $\{-2, -1, 0, 1, 2\}$
in a random minimal factorization of size $n = 500$.

Local convergence of trajectories (2/3)

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and look around $i = 0$:

Theorem (F., Kortchemski)

There is a family $(X_i)_{i \in \mathbb{Z}}$ of integer-valued step functions on $[0,1]$

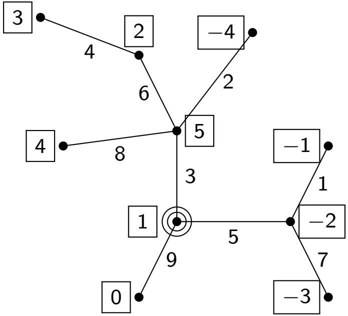
$$\left(X_i^{(n)}(\lfloor nt \rfloor) : 0 \le t \le 1 \right)_{i \in \mathbb{Z}} \xrightarrow[n\to\infty]{(d)} (X_i)_{i \in \mathbb{Z}}$$

holds in distribution (in the sense of convergence of finite dimensional marginals).

Note: there is a rescaling in time, but not in space.

Local convergence of trajectories (3/3)

For this we need to study the local convergence around 1 of the edge and vertex-labelled tree of the factorization.



Tree of the factorization
(-1,-2), (-4, 5), (1, 5), (2, 3), (-2, 1), (2, 5), (-3,-2), (4, 5), (0, 1)

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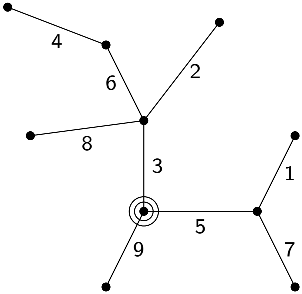

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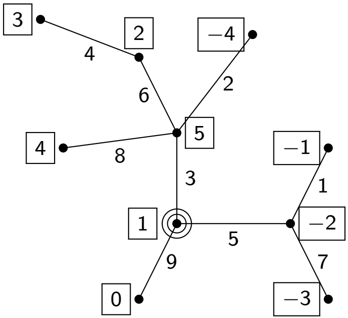

- Standard: the edge-labelled tree converges to "Kesten tree".
- Difficulty: understand the vertex labelling and prove that it's "local".

Conclusion

1. Global and local convergence results for uniform random factorizations of a long cycle into transposition.

2. For the global convergence, we are missing the convergence of the process of non-crossing partitions, we only have convergence of the marginals.

3. Factorization models have a very rich combinatorics (Hurwitz numbers, maps, genomics), but there are almost no probabilistic results. There is work to do...

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1. Global and local convergence results for uniform random factorizations of a long cycle into transposition.
2. For the global convergence, we are missing the convergence of the process of non-crossing partitions, we only have convergence of the marginals.
3. Factorization models have a very rich combinatorics (Hurwitz numbers, maps, genomics), but there are almost no probabilistic results. There is work to do...

Thank you for your attention