Random representations of the symmetric groups

I Notations and statement of the problem

1. Representation theory of finite groups.

<u>Def</u>: A repr (ρ, V) of a finite group G is: $\boxed{k = \mathbb{C}}$

- a finite dim $\mathbb{C}$-v.s. V
- a morphism $\rho : G \to GL(V)$

The character of ρ is the function $\chi^\rho : G \to \mathbb{C}$

$$\chi^\rho(g) = \mathrm{tr}(\rho(g))$$

<u>Thm</u>: . Every repr. of G is a direct sum of <u>irred.</u> representations (+ unicity) $\rightarrow$ we study only irred. rep.

. For a given group G, there are finitely many ir. rep. (in fact, as many as conjugacy classes in G)

<u>Notation</u>: $\mathcal{Y}_G$ index set of ir. rep. of G

λ typical elt of $\mathcal{Y}_G$ $\rightsquigarrow$ $V^\lambda, \rho^\lambda, \chi^\lambda, \hat{\chi}^\lambda := \frac{\chi^\lambda}{\dim(V^\lambda)}$

2. Plancherel measure

<u>Def</u>: $\mathbb{P}_G$ is the probability measure on $\mathcal{Y}_G$ defined by

$$\mathbb{P}_G(\lambda) = \frac{(\dim V_\lambda)^2}{|G|}$$

<u>Rk</u>: $\sum_{\lambda \in \mathcal{Y}_G} \mathbb{P}_G(\lambda) = 1$ as implicitely claimed

because $\underset{\substack{\uparrow \\ \text{left reg. repr.}}}{\mathbb{C}[G]} = \bigoplus_{\lambda \in \mathcal{Y}_G} V_\lambda^{\dim(V_\lambda)}$

Prop: Fix $g \in G$

$$\mathbb{E}_{P_G}\left(\hat{\chi}^{\bullet}(g)\right) = \begin{cases} 1 & \text{if } g = id_G ; \\ 0 & \text{else.} \end{cases}$$

$\lambda \mapsto \hat{\chi}^{\lambda}(g)$ is a random variable on $\mathcal{Y}_G$.

Proof:

$$\underbrace{tr_{\mathbb{C}[G]}(g)}_{= \delta_{id,g} \cdot |G|} = \sum_{\lambda \in \mathcal{Y}_G} \underbrace{(\dim(V_\lambda))\, \chi^{\lambda}(g)}_{= E_{P_G}(\hat{\chi}^{\bullet}(g))}$$

Rk: This proposition characterizes Plancherel measure
We will only use this fact!

3. Our problem

- Take a sequence of group $(G_n)_{n \geq 1}$. Assume $G_1 \subseteq G_2 \subseteq \ldots$
- For each n, let $\lambda^{(n)} \in \mathcal{Y}_{G_n}$ random representation (distrib: Plancherel)

Question: describe the asymptotic behaviour of $\lambda^{(n)}$.

$\rightarrow$ vague: we don't know in general where $\lambda^{(n)}$ lives...

a precise related question:

fix $g \in G_k$. Describe asymp. of $\left(\hat{\chi}^{\lambda^{(n)}}(g)\right)_{n \geq k}$

4. Symmetric groups

Here, we focus on the case $G_n = S_n$;

$\rightarrow$ ext. to B_n (D_n? $G_{n,d,e}$?) should not be too hard;

$\rightarrow$ some results exist for $GL_n(\mathbb{F}_q)$, q fixed.

$\rightarrow$ dream: a general unified theorem for groups "whose elements are functions" well-known for S_n : $\mathcal{Y}_{S_n} = \{$ "partitions" of $n\}$ + $\boxed{\text{numerous formulas for character values}}$

Young diagrams of size n

- ~~we do not need them~~ to answer the question about $\hat{\chi}^{\cdot}(g)$!
- possible to define a notion of "cv of Young diagrams" and to prove asympt. results for $\lambda^{(n)}$. (3rd talk)

II Moment method

1. Preliminaries : group algebra and its center

Def : $\cdot$ group algebra of G is

$$\mathbb{C}[G] := \left\{ \sum_{g \in G} c_g\, g \,,\ c_g \in \mathbb{C} \right\}$$ i.e formal linear combination of elements of G

product by distributivity $\Rightarrow$ algebra structure.

The center is $Z(\mathbb{C}[G]) := \{ x \in \mathbb{C}[G] : \forall y \in \mathbb{C}[G],\ xy = yx \}$
$= \mathrm{Vect}(K_\mu)_{\mu \text{ conj class of } G}$

where $K_\mu := \sum_{g \text{ in the conj class } \mu} g$

Obs : repr of $G \Longleftrightarrow (\rho, V)$ with
- V ....
- ρ unital algebra morphism $\mathbb{C}[G] \to \mathrm{End}(V)$

in other terms, we extend ρ, χ^ρ, $\hat{\chi}^\rho$, ... to the group algebra by linearity.

Lemme (Schur) : Let $x \in Z(\mathbb{C}[G])$ and $\lambda \in \mathcal{Y}_G$. Then
$$\rho^\lambda(x) = \hat{\chi}^\lambda(x) \cdot \mathrm{id}_{V_\lambda}$$

Corollary : Let $x \in Z(\mathbb{C}[G])$, $y \in \mathbb{C}[G]$ and $\lambda \in \mathcal{Y}_G$.
$$\hat{\chi}^\lambda(xy) = \hat{\chi}^\lambda(x)\, \hat{\chi}^\lambda(y).$$

2. Moments of $\hat{\chi}^{\bullet}((1,2))$. (Réf. Hora, Comm. Math. Phys., 1998)

$$\mathbb{E}_n\left(\hat{\chi}^{\bullet}((1,2))\right) = 0 \quad \text{(prop from intro)}$$

short not. for $\mathbb{E}_{\mathbb{P}_{S_n}}$

$$\mathbb{E}_n\left(\hat{\chi}^{\bullet}((1,2))^2\right) = \ ?$$ ← difficulty: $\mathbb{E}$ is not multiplicative but $\hat{\chi}$ is multiplicative on the center!

$\Rightarrow$ Write $\hat{\chi}^{\lambda}((1,2)) = \frac{2}{n(n-1)} \hat{\chi}^{\lambda}\left(K_{(2,1^{n-2})}\right)$

$K_{(2,1^{n-2})}$: sum of all transpo in S_n

By Schur lemma, $\hat{\chi}^{\lambda}\left(K_{(2,1^{n-2})}\right)^2 = \hat{\chi}^{\lambda}\left(K_{(2,1^{n-2})}^2\right)$

But $K_{(2,1^{n-2})}^2 = 2\, K_{(2,2,1^{n-2})} + 3\, K_{(3,1^{n-3})} + \binom{n}{2} K_{(1^n)}$, with $K_{(1^n)} = \mathrm{id}_n$

$$\Rightarrow \mathbb{E}_n\left(\hat{\chi}^{\bullet}((1,2))^2\right) = \frac{4}{n^2(n-1)^2}\, \mathbb{E}\left(\hat{\chi}^{\bullet}\left(2K_{(2,2,1^{n-2})} + 3K_{(3,1^{n-3})} + \binom{n}{2} K_{(1^n)}\right)\right)$$

exp. is zero.

$$= \frac{2}{n(n-1)}.$$

Obs: only coef of identity is important

In general,

$$\mathbb{E}_n\left(\hat{\chi}^{\bullet}((1,2))^m\right) = \binom{n}{2}^{-m} \underbrace{\#\text{ factorisations of } \mathrm{id}_n \text{ as a product of } m \text{ transpositions}}_{F_n^m}$$

($= 0$ if m is odd)

(no close formula)

3. Asymptotics

$$F_n^{2m} = \frac{1}{2^{2m}} \#\left\{ (i_1, j_1, \dots, i_{2m}, j_{2m}) : \begin{array}{l} (i_1, j_1) \cdots (i_{2m}, j_{2m}) = \mathrm{id} \\ i_1 \neq j_1, \dots, i_{2m} \neq j_{2m} \end{array} \right\}$$

To a list $(i_1, j_1, \dots, i_{2m}, j_{2m})$, we associate a set partition $\mathrm{Ker}(\underline{i}, \underline{j})$ of $\{1, 1'; \dots; 2m, 2m'\}$ such $t \sim s \Leftrightarrow i_t = i_s$.

Obs: condition $(i_1, j_1) \cdots (i_{2m}, j_{2m}) = id$
$i_1 \neq j_1 \cdots i_{2m} \neq j_{2m}$ depends only on $\ker(\underline{i}, \underline{j})$

$\Rightarrow$ Call a set-partition Π good if conditions are fulfilled.

$$2^{2m} F_n^{2m} = \sum_{\pi \text{ good}} \# \{ (i_1, j_1 \dots) : \operatorname{Ker}(\underline{i}, \underline{j}) = \pi \}$$
$$= \sum_{\pi \text{ good}} (n)_{\ell(\pi)}$$

$\rightarrow$ We are interested in $n \to \infty$ asympt.

$\Rightarrow$ What are the good set-partitions with maximal length?

Answer

- clearly $\ell(\pi) \leq 2m$ (each number must appear at least twice)
- $\ell(\pi) = 2m \Rightarrow$ each transposition appear exactly twice

$$2^{2m} F_n^{2m} = \underset{\substack{\uparrow \\ \text{matching}}}{(2m-1)!!} \cdot \underset{\substack{\uparrow \\ \text{for each pair of equal transpo} \\ \text{either } i_t = i_s \text{ and } j_t = j_s \\ i_t = j_s \text{ and } i_s = j_t}}{2^m} \cdot n^{2m} \left(1 + O(1/n)\right)$$

$$\Rightarrow \mathbb{E}_n\left(\hat{\chi}^{\cdot}((1,2))^{2m}\right) = (2m-1)!!\, 2^m\, n^{-2m} \left(1 + O(1/n)\right)$$

$$\frac{n\, \hat{\chi}^{\cdot}((1,2))}{\sqrt{2}} \longrightarrow \mathcal{N}(0,1)$$

4. Comments

Key lemma: in facts which contribute to asympt, cycles are pairwise inv. to each other

- Easy gen: $n^{k/2}\, \hat{\chi}^{\cdot}((1, \dots, k)) / \sqrt{k} \longrightarrow \mathcal{N}(0,1)$
  + asympt. independence for different k's
- Harder: cv of $n^{k/2}\, \hat{\chi}^{\cdot}(\sigma) \xrightarrow{d} \dots\dots$ for $\sigma \in S_k$ w/t fixed points.
  subtle combinatorics

• bounds on speed of cv via Stein method. $O(n^{-1/4})$
(Fulman 2003, F., Dołęga 2014)

III Partial permutations and polynomial functions

1. Partial permutations

Goal: understand better relations as

$$K_{(2,1^{n-2})}^2 = \binom{n}{2} K_{(1^n)} + 2K_{(2,2,1^{n-2})} + 3K_{(3,1^{n-3})}$$

$$K_{(2,1^{n-2})}^{2m} = (2m-1)!!\, 2^{-m} n^{2m} (1+O(1/n)) K_{(1^n)} + \text{other terms}$$

Def: A partial permutation is a pair (d, σ)
where • $d \subset \mathbb{N}^*$ finite set
• $\sigma : d \to d$ bijection

equivalently, perm. with two type of fixed points

Not: $(\{1;2;5\}, (1;5)) = (1,5)(2) \neq (1,5)(3) = (\{1;3;5\}, (1;5))$

Product: $(d,\sigma), (d',\sigma') = (d \cup d', \tilde{\sigma} \circ \tilde{\sigma}')$

$\sim$ means that we extend perm with fixed points (here to $d \cup d'$)

def: $\mathcal{B}_\infty$ = algebra of infinite linear comb of (σ, d)

Obs: $\varphi_n : \mathcal{B}_\infty \longrightarrow \mathbb{C}[S_n]$

$$(d,\sigma) \longmapsto \begin{cases} \tilde{\sigma} & \text{if } d \subseteq \{1,\dots,n\} \\ 0 & \text{otherwise} \end{cases}$$

extension to $\{1,\dots,n\}$

is an algebra morphism.

without d, we could not construct such a morphism.

Let μ be an integer partition. Set

$$A_\mu = \sum_{\substack{d \subseteq \mathbb{N} \\ \sigma: d \to d \\ \text{type } \mu}} (d,\sigma)$$

ex: $A_{(2)} = \sum_{i<j} (i,j)$

$A_{(2,1)} = \sum_{\substack{i<j \\ k \neq i,j}} (i,j)(k)$

Prop: $\mathrm{Vect}(A_\mu)_{\mu \text{ partitions of } \underline{\text{all}} \text{ sizes}}$ is a subalgebra of $\mathcal{B}_\infty$

finite l.c.

Proof: elements $\sum_{(d,\sigma)} c_{(d,\sigma)} (d,\sigma) \in \mathrm{Vect}(A_\mu)$ are exactly elts of $\mathcal{B}_\infty$ s.t.:

can be skipped

- inv. by action of S_n $\tau \cdot (d,\sigma) = (\tau(d), \tau\sigma\tau^{-1})$
- $\{|d| ; \exists \sigma, c_{(d,\sigma)} \neq 0\}$ finite $\square$

Prop: (A_k) alg basis of $\mathrm{Vect}(A_\mu)$.

Proof: $A_{k_1} \cdots A_{k_r} = A_{(k_1,\dots,k_r)} + \sum_{|\nu| < k_1+\dots+k_r} * A_\nu$ $\square$

Write $A_\mu \cdot A_\nu = \sum_{\rho} c_{\mu,\nu}^{\rho} A_\rho$

ex $A_{(2)} \cdot A_{(2)} = 2A_{(2,2)} + 3A_{(3)} + A_{(1^2)}$

$\Rightarrow K_{(2,1^{n-2})} \cdot K_{(2,1^{n-2})} = 2K_{(2,2,1^{n-4})} + 3K_{(3,1^{n-3})} + \binom{n}{2} K_{(1^n)}$

$\rightsquigarrow$ always pol. coef in this kind of expressions (Farahat, Higman 1957; Kerov, Ivanov 1999)

(Gen in progress by O. Tout 2014+)

2. Polynomial functions

A_μ can be seen as a sequence $(a_n)_{n\geq 1}$, $a_n \in Z(\mathbb{C}[S_n])$

$(a_n = \varphi_n(A_\mu))$

Recall: $\hat{\chi}$ defines an algebra isomorphism

(discrete Fourier transform) $Z(\mathbb{C}[G]) \longrightarrow \mathcal{F}(\mathcal{Y}_G, \mathbb{C})$

$x \longmapsto (\lambda \mapsto \hat{\chi}^\lambda(x))$

set of all Young diagrams

Consequence $\mathrm{Vect}(A_\mu) \subseteq \mathcal{F}(\sqcup \mathcal{Y}_{S_n}, \mathbb{C})$

$A_\mu \longmapsto (\lambda \mapsto \hat{\chi}^\lambda(\varphi_n(A_\mu))) =: \mathrm{Ch}_\mu / z_\mu$

where $$ch_\mu(\lambda) = \begin{cases} \dfrac{n(n-1)\cdots(n-k+1)}{z_\mu} \hat{\chi}^\lambda(\mu\, 1^{n-k}) \\ 0 \end{cases}$$

Obs: $Ch_{(1)}(\lambda) = |\lambda|$

Clearly, $\mathrm{Vect}(ch_\mu)$ is a subalgebra of $\mathcal{F}(\sqcup \mathcal{Y}_{S_n}, \mathbb{C})$ called "<u>polynomial functions on Young diagrams</u>" denoted Λ^*

(Kerov, Olshanski, 1994)

Note: multiplication is easier to understand on partial permutations.

Prop: $Ch_{(k)}$ is an algebraic basis of Λ^*.

3. Filtrations and asymptotics

Prop: let $I \subseteq \mathbb{N}$. Then

$$\deg_I((\sigma, d)) = |d| + \sum_{i \in I} \underbrace{\#\text{cycles of size } i \text{ in } \sigma}_{m_i(\sigma)}$$

defines a filtration on $\mathcal{B}_\infty$.

Sketch of proof: Assume $(\sigma, A) \cdot (\tau, B) = (\pi, C)$

$(C = A \cup B,\ \pi = \tilde{\sigma} \circ \tilde{\tau})$

We have to prove

$$\sum_{i \in I} m_i(\pi) - m_i(\sigma) - m_i(\tau) \leq \underbrace{|A| + |B| - |C|}_{|A| \cap |B|}$$

But, each "new" cycle of π contains at least one element in $A \cap B$.

neither a cycle of σ
nor a cycle of τ

Corollary: $\deg_I(Ch_\mu) = |\mu| + \sum_{i \in I} m_i(\mu)$ defines a filtration on Λ^*

Application to asymptotics

Lemma: Let $1 \in I$ and $F \in \Lambda^*$.
Then $\mathbb{E}_n(F) = O\left(n^{\deg_I(F)/2}\right)$

$F/\mathcal{Y}_n$ is a r.v.

Proof: true on a basis ($F = Ch_\mu$)

Corollary: if $m > \deg_I(F)/2$ (F, I as above)
$$F(\lambda)/n^m \xrightarrow{P} 0$$

method that we shall use to study F asympt.

- first order: look at the top degree term of F in the $(Ch_{(k)})$ alg. basis (for $\deg_{\mathbb{N}}$) and use
$$\frac{Ch_{(k)}}{n^{\deg_{\mathbb{N}}/2}} \xrightarrow{P\,(a.s.)} \delta_{k,1}$$
- second order: idem with $\deg_{\{1\}}$ and
$$\text{for } k \geq 2, \quad \frac{Ch_{(k)}}{n^{\deg_{\{1\}}/2}} \longrightarrow \sqrt{k}\, \mathcal{N}(0,1)$$
(+ asymp ind.)

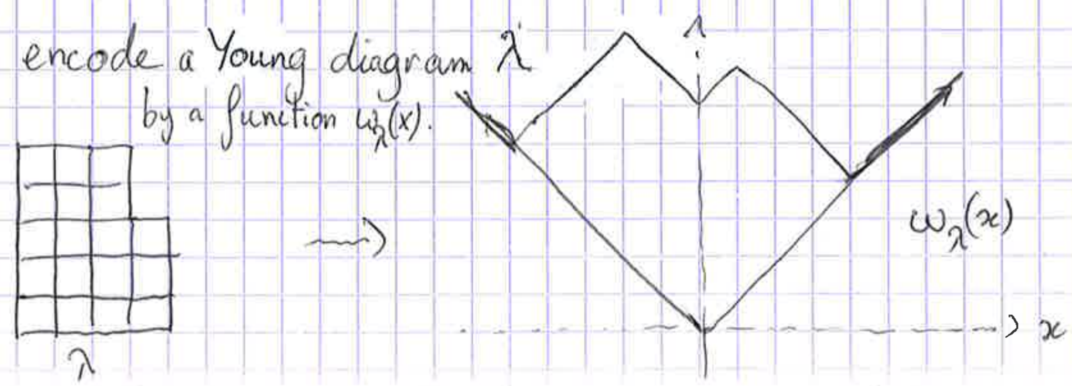
IV Geometric convergence

1. Convergence of Young diagrams

We now want to study the sequence $(\lambda^{(n)})$, not $(\hat{\chi}^{\lambda^{(n)}}(\sigma))$

$\rightarrow$ What does it mean that $\lambda^{(n)}$ converges?

Idea: encode a Young diagram λ by a function $\omega_\lambda(x)$.



$$\{\text{Young diagrams}\} \subseteq \left\{\begin{array}{l}\text{continuous 1-Lipschitz function} \\ \text{equal to } |x| \text{ outside a compact interval}\end{array}\right\}$$

called "continuous Young diagrams"

Two notions of conv. of this space

- "strong" conv.:

$$\| \omega_n(x) - \omega(x) \|_\infty \longrightarrow 0$$

(ou L^p)

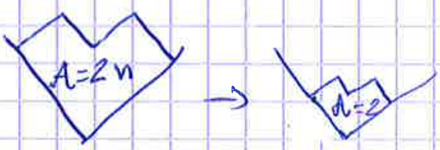
- "weak" conv.:

for $k \geqslant 0$, $\displaystyle\int_{-\infty}^{\infty} (\omega_n(x) - \omega(x))\, x^k\, dx \longrightarrow 0$

Note: in our context, it is necessary to renormalize

$$\omega_{\overline{\lambda^{(n)}}}(x) := \frac{1}{\sqrt{n}}\, \omega_\lambda(t\sqrt{n})$$

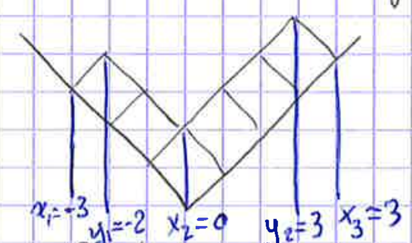
dilatation of both axes to have cst area (here, 2)



2. A formula for character values on cycles.

We know that $\chi^{\lambda^{(n)}}(\cdot)$ cv, we want to prove that $\omega_{\lambda^{(n)}}$ cv

$\rightarrow$ we need a formula to link both quantities.



$\lambda = (4,1,1)$

$$G_\lambda(z) = \frac{\prod_{i=1}^{m-1} (z - y_i)}{\prod_{i=1}^{m} (z - x_i)}$$

COORDONNÉES ENTRELACÉES (Kerov, 1996)

$$H_\lambda(z) = 1 / G_\lambda(z)$$

careful analysis of $\chi^\lambda(1,\dots,k)$ $= [s_\lambda] p_k p_1^{n-k}$ with det expression of s_λ.

$$\rightarrow Ch_k(\lambda) = -\frac{1}{k} [z^{-1}]\; H_\lambda(z) \cdots H_\lambda(z-k+1)$$

coef of z^{-1} in the expansion at $z = \infty$ (negative powers)

(Frobenius, 1900, Macdonald, 1995)

3. Functions $\tilde{p}_k$ and their asymptotics

- Define $\log(z\, G_\lambda(z)) = \sum_{k\geq 1} \frac{\tilde{p}_k(\lambda)}{k} z^{-k}$

Rq: $\tilde{p}_1(\lambda) = 0 \ \forall \lambda$.

formula above implies

$$Ch_k = \frac{\tilde{p}_{k+1}}{k+1} + \text{polynom in } \tilde{p}_2, \tilde{p}_3, \dots, \tilde{p}_{k-1}$$

Corollary: $(\tilde{p}_k)_{k\geq 2}$ is an algebraic basis of $Vect(Ch_\mu)$.

Obs: $\tilde{p}_k(\lambda) = \sum_{i=1}^{m} x_i^k - \sum_{i=1}^{m-1} y_i^k = \frac{k(k-1)}{2} \int_{-\infty}^{\infty} x^{k-2} (\omega_\lambda(x) - |x|)\, dx$

informally $\frac{1}{2} \int_{-\infty}^{\infty} x^k (\omega_\lambda(x) - |x|)'' dx$

Thus weak convergence $\Longleftrightarrow$ convergence of the $\tilde{p}_k(\lambda)$.

$\rightarrow$ we need the top-component expr. of $\tilde{p}_k$ in $(Ch_{(\rho)})$ basis

Prop: $$\tilde{p}_k = \sum_{\substack{m_2, m_3, \dots \\ 2m_2 + 3m_3 + \dots = k}} \frac{(k)_{\sum m_i}}{\prod m_i!} \prod_{i\geq 2} \left(Ch_{(i-1)}\right)^{m_i} + \text{smaller terms for degree}$$

and $$\tilde{p}_k = \begin{cases} (2m)!/m!m! \; Ch_{(1)}^m & k=2m \\ 0 & k \text{ odd} \end{cases} + k \sum_{j=0}^{\lfloor k-3/2 \rfloor} \binom{k-1}{j} Ch_{(k-1-2j)} Ch_1^j + \text{sm. terms for } \deg_{\{1\}}$$

$$\Rightarrow \tilde{p}_k(\lambda^{(n)}) = \frac{p_k(\lambda)}{n^k} \xrightarrow[\text{or a.s.}]{(P)} \begin{cases} (2m)!/m!m! & \text{if } k = 2m \\ 0 & \text{if } k \text{ odd} \end{cases}$$

$\Rightarrow$ $\tilde{p}_k$ Gaussian fluctuation with explicit cov limits

$$\lim_{n\to\infty} n \operatorname{Cov}\left(\tilde{p}_{k_1}(\lambda^{(n)}), \tilde{p}_{k_2}(\lambda^{(n)})\right) = \sum_{\substack{j_1, j_2 \\ 2j_1 \leq k_1 - 3 \\ 2j_2 \leq k_2 - 3 \\ k_1 - 1 - 2j_1 = k_2 - 1 - 2j_2}} k_1 k_2 \binom{k_1 - 1}{j_1} \binom{k_2 - 1}{j_2}$$

4. What is the limit shape?

$$\Omega(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin \frac{x}{2} + \sqrt{4-x^2}\right) & \text{for } |x| \leq 2 \\ |x| & \text{for } |x| > 2 \end{cases} \rightsquigarrow \Omega'(x) = \frac{2}{\pi} \arcsin \frac{x}{2}$$

Exercise: check that $\tilde{p}_k(\Omega) = \begin{cases} (2m)!/m!\,m! & \text{if } k = 2m; \\ 0 & \text{if } k \text{ odd.} \end{cases}$

How to guess the formula for Ω:

- using $K(z) = G^{\langle -1 \rangle}(z)$ and free cumulants
  see Biane 2001
- direct way by computing its Fourier transform?
  I don't know.

5. Strong convergence

We know for each polynomial P,

$$\int_{-\infty}^{\infty} \omega_{\overline{\lambda_n}}(x)\, P(x)\, dx \to \int_{-\infty}^{+\infty} \Omega(x)\, P(x)\, dx.$$

want to prove $\quad \| \omega_{\overline{\lambda_n}}(x) - \Omega(x) \|_\infty \to 0$

on a compact interval, that would be easy:

Weierstrass $\Rightarrow \displaystyle\int_I \omega_{\overline{\lambda_n}}(x)\, f(x)\, dx \to \int_I \Omega(x)\, f(x)\, dx$
for f continuous

$f =$ 

$+$ 1-Lipschitz $\Rightarrow \omega_{\overline{\lambda_n}}(x_0) \longrightarrow \Omega(x_0) \quad \forall x_0 \in I$

1-Lipschitz $\Rightarrow \| \omega_{\overline{\lambda_n}} - \Omega \|_\infty \longrightarrow 0$

But $\mathbb{R}$ not compact. However, one can show that, with high proba $\quad \mathrm{Supp}\left(\omega_{\overline{\lambda_n}}(x) - |x|\right) \subseteq [-3;3]$

Lemme: λ random Young diagram with $\mathbb{P}_n$.
Then $\mathbb{P}(\lambda_1, \lambda'_1 \leq 3\sqrt{n}) \longrightarrow 1$

Proof: $\mathbb{P}(\lambda_1 \geq 3\sqrt{n}) = \mathbb{P}$(uniform rand perm has an incr. subsequence of length $3\sqrt{n}$)

$\leq \mathbb{E}$(# incr subs of length $3\sqrt{n}$ in a unif rand perm.)

$$\leq \binom{n}{3\sqrt{n}}^2 (n-3\sqrt{n})!$$

... (Stirling)

$$\leq o(1).$$

Conclusion: $\|\omega_{\overline{\lambda_n}} - \Omega\|_\infty \longrightarrow 0$

(Kerov, Vershik / Logan, Shepp 1977)
(this proof: Kerov, Ivanov, Olshanski, 2002)