A simple model of trees for unicellular maps

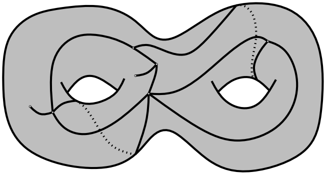
Valentin Féray
joint work with Guillaume Chapuy (LIAFA) and Éric Fusy (LIX)
All the figures of this talk have been made
either by Guillaume (maps) or by Éric (trees), thanks to them.

LaBRI, CNRS, Bordeaux

Groupe de travail
Combinatoire énumérative et algébrique
Bordeaux, 13 janvier 2012

Summary

- We consider unicellular maps. Counting this object is a problem related to matrix integrals, symmetric group algebra, ...



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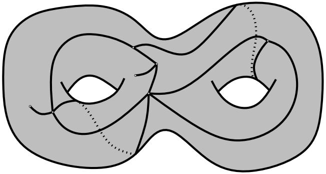

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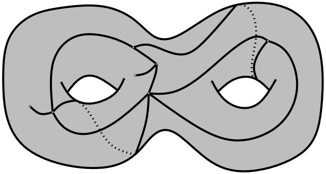

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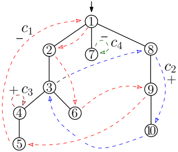

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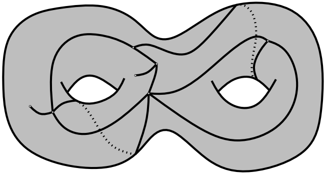

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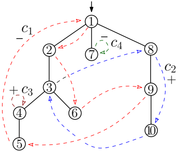

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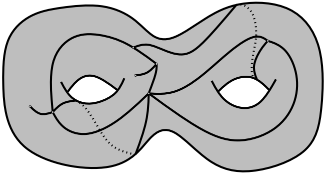

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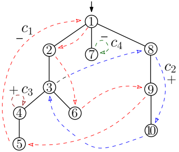


- Trees = "recursive structure" ⇒ easy to handle
  ⇒ combinatorial proofs of a lot of formulas.

- The bijection converses a lot of structure,
  but is not explicit.

Outline of the talk

1. Presentations of the objects

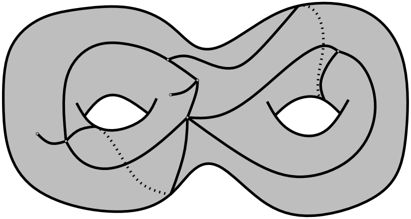
2. Existence of a bijection

3. Combinatorial proofs of enumerative formulas for maps

What is a (rooted) unicellular maps?

3 equivalent descriptions

A graph drawn of a 2-dimension surface such that *the complementary is homeomorphic to an open disc.*



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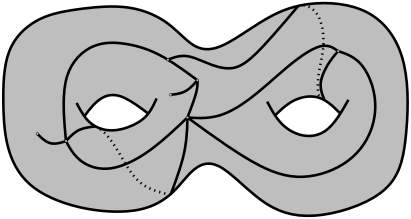

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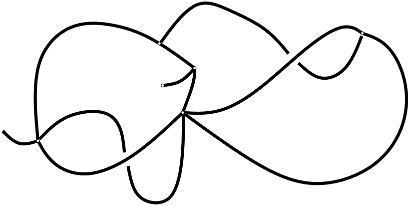
A graph with a cyclic order of edges around vertices such that *there is only one face*



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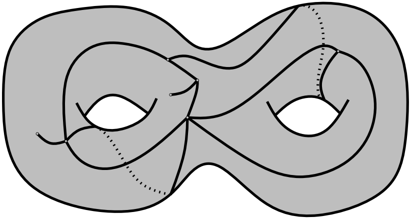

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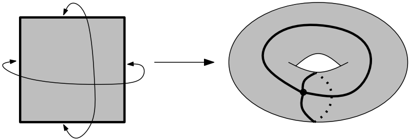
A merging of the edges of a polygon



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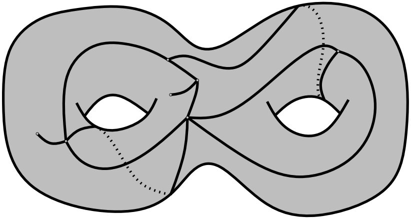

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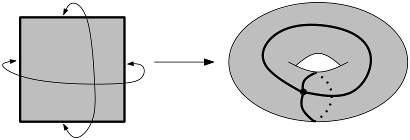


There are $(2n-1)!!$ rooted unicellular maps with n edges.

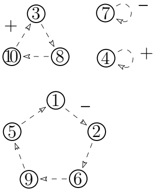
But it is hard to count them with a prescribed genus (or number of vertices).

C-permutation

Definition

A C-permutation of size $n+1$ is a permutation of $n+1$

- with only cycles of odd lengths;
- with the additional data of a sign for each cycle.



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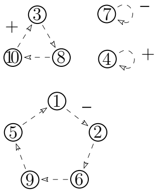
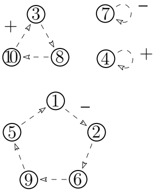


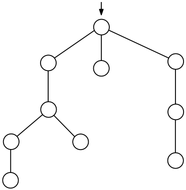
genus: $g := 1/2(n+1-\#(\text{cycles}))$.

Decorated trees

Definition

A C-decorated tree is a couple (T, σ), where T is a tree with $n+1$ vertices and σ a C-permutation of size $n+1$.





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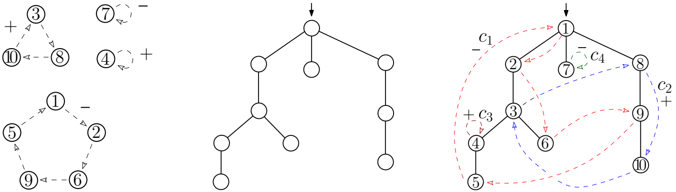
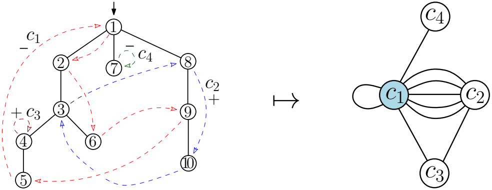


We can see the permutation as acting on the vertices of the tree by doing (for instance) a left-to-right depth-first traversal.

Underlying graph

If we merge the vertices in the same cycles, we get a (rooted) graph called *underlying graph* C-decorated tree.



Statement of the main result

Notations:
$\mathcal{E}_g(n)$ set of maps with n edges of genus g
$\mathcal{T}_g(n)$ set of C-decorated trees with n edges of genus g

Theorem

There is a bijection

$$\left[2^{n+1}\right] \times \mathcal{E}_g(n) \simeq \mathcal{T}_g(n)$$

which preserves the underlying graphs.

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Proof.

True for $g = 0$.
We will see that the two sets fulfill the same induction relation on g.
For unicellular maps, we use a previous construction of G. Chapuy. $\square$

Chapuy's bijection (1/2)

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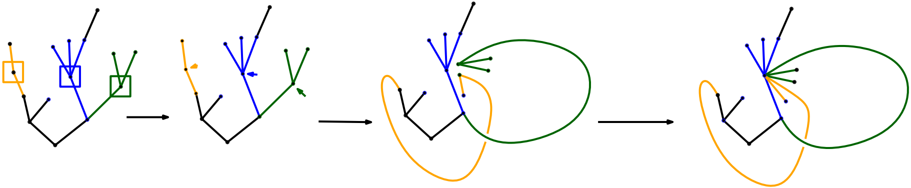

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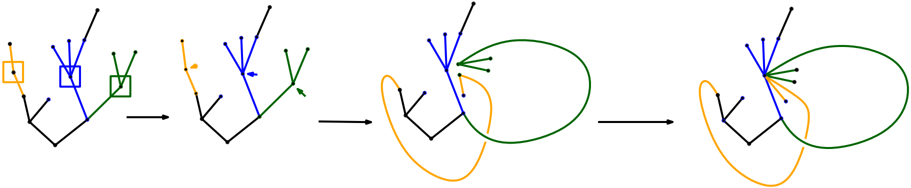

⚠ Mergings in maps are not well-defined (a lot of choices to do).
The *unicellular* condition is not necessarily preserved when we glue 3 vertices ($\rightarrow$ quite technical construction).

Chapuy's bijection (2/2)

This defines a map

$$\mathcal{E}_g^{(3)}(n) \longrightarrow \mathcal{E}_{g+1}(n),$$

where $\mathcal{E}_g^{(3)}(n)$ set of maps with n edges of genus g *with 3 marked vertices*.

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Theorem (Chapuy, 2009)

for $g > 0$ *and* $n \geq 0$,

$$[2g] \times \mathcal{E}_g(n) \simeq \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \mathcal{E}_{g-3}^{(7)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n). \qquad (1)$$

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In addition, if M and (M', S') are in correspondence, then the underlying graph of M is obtained from the underlying graph of M' by merging the vertices in S' into a single vertex.

A variant of Foata fundamental transform

Lemma

There is a bijection

$$\varphi : S_n \times \{-;+\} \simeq \{C\text{-permutation of } n\}.$$

Description of the bijection on the example $(4371562, -)$.

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Problem: it has even size! We move the second element (here 5) and record that operation with a $-$ sign: we get

$$4375, \;\; {}^{-}(162).$$

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We cut the word at its new minimum (here 3):

$$4|375, \ {}^{-}(162).$$

The second part is of odd size, we can consider it as a cycle of the C-permutation (we assign a $+$ sign to it).

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The first cycle has always a $+$ sign! We assign to it the sign of the input instead.

$$\varphi\big((4371562, -)\big) = \;{}^{-}(4)\; {}^{+}(375)\; {}^{-}(162)$$

Decomposition of C-permutations

Take a C-permutation with a marked element.

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This is invertible!

Decomposition of C-decorated trees

Notations

$\mathcal{P}_g(n)$: set of C-permutations of genus g and size n.
$\mathcal{P}_g^{(k)}(n)$: idem with k marked cycles.

Corollary

There is a bijection

$$\varphi : [n+1] \times \mathcal{P}_g(n+1) \simeq \mathcal{P}_g^{(1)}(n+1) \sqcup \mathcal{P}_{g-1}^{(3)}(n+1) \sqcup \cdots \sqcup \mathcal{P}_0^{(2g+1)}(n+1).$$

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Moreover, the partition into cycles of x is obtained by merging the marked cycles in the partition of $\varphi(x)$.

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But $\mathcal{T}_g(n) = \{\text{arbres à } n \text{ sommets}\} \times \mathcal{P}_g(n+1)$.

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Moreover, the underlying graph of x is obtained by merging the marked vertices in the underlying graph of $\Psi(x)$.

End of the proof

- Suppose that for all $g' < g$, there exists a underlying graph preserving bijection
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with a bijection preserving underlying graphs.
- One has to extract a bijection from the $2g$-to-$2g$ correspondence above. This can be done using Hall marriage theorem (☹not explicit!).

Lehman-Walsh formula

Theorem (Lehman and Walsh, 1972)

The number of maps of genus g with n edges is given by

$$|\mathcal{E}_g(n)| = \frac{(2n)!}{n!(n+1-2g)!2^{2g}} \sum_{\gamma \vdash g} \frac{(n+1-2g)_\ell}{\prod_i m_i!(2i+1)^{m_i}},$$

where $(x)_k = x(x-1)\dots(x-k)$, ℓ is the number of parts of γ, and m_i is the number of parts of length i in γ.

Proof.

Look at the possible cycle types of a C-permutation of genus g. Details on the white board. □

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Remark: this is the first combinatorial proof of this formula. We obtain combinatorial proofs of a lot of formulae in a unified way (Harer-Zagier recurrence, Jackson summation, Goupil-Schaeffer formulae)

Open problems

- Find an explicit bijection.
- Read “some” information on the rotation system on the tree model, for example to count constellations (Poulhalon-Schaeffer formula).