Polynomial functions on Young diagrams and limit shape of Young diagrams

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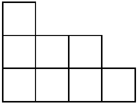
Seminar über stochastische Prozesse
Universität Zürich, ETH

What is this talk about?

- Young diagrams $\lambda \vdash n$: 

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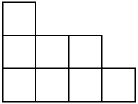

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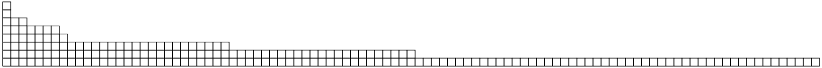

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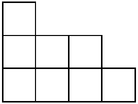

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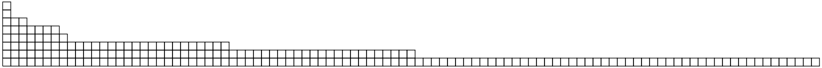

- Question: asymptotic behaviour of the lengths of its first rows, its first columns, ...
- Tool: representation theory of symmetric groups.

Outline of the talk

1. Model of random Young diagrams
2. Polynomial functions on Young diagrams
3. How to prove asymptotic results?
4. A few remarks

What is a representation?

S_n: group of permutations of elements $1, 2, \cdots, n$.

A representation of S_n is a couple (V, ρ) where:

- V is a finite dimensional $\mathbb{C}$ vector space;
- ρ is a morphism $S_n \to GL(V)$.

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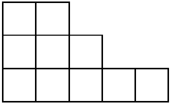
We are interested in irreducible representations.

Theorem

Irreducible representations of S_n are in bijection with partitions $\lambda \vdash n$ or, equivalently, Young diagrams λ with n boxes.

Ex: $n = 10$,
$\lambda = (5, 3, 2)$.



Characters

Let (V, ρ) be a representation of S_n. We define:

$$\chi^\rho(\sigma) = \frac{\mathrm{Tr}\left(\rho(\sigma)\right)}{\dim(\lambda)} \text{ for } \sigma \in S_n.$$

It i s the (normalized) character of ρ.

No loss of information!

$$(\rho, V) \simeq_{S_n} (\rho', V') \Leftrightarrow \chi^\rho = \chi^{\rho'}.$$

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Indeed,

$$\begin{aligned}\chi^\rho(\tau^{-1}\sigma\tau) &= \frac{\mathrm{Tr}\left(\rho(\tau^{-1}\sigma\tau)\right)}{\dim(\lambda)} \\ &= \frac{\mathrm{Tr}\left(\rho(\tau)^{-1}\cdot\rho(\sigma)\cdot\rho(\tau)\right)}{\dim(\lambda)} = \frac{\mathrm{Tr}\left(\rho(\sigma)\right)}{\dim(\lambda)}\end{aligned}$$

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Theorem

The characters χ^λ of irreducible representations ρ^λ of the symmetric group S_n form an (orthogonal) basis of the space of central functions on S_n.

Some central functions on $\bigcup S_n$

Let $\omega = (\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots, \gamma)$ with

- α and β infinite non-increasing sequences;
- $\alpha_i, \beta_i, \gamma \geq 0$;
- $\sum_i \alpha_i + \sum_i \beta_i + \gamma = 1$.

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Define

$$\begin{aligned} p_1(\omega) &= 1 \\ p_k(\omega) &= \sum a_i^k + (-1)^{k-1} \sum b_i^k \\ p_{(\mu_1, \mu_2, \dots)}(\omega) &= \prod_j p_{\mu_j}(\omega). \end{aligned}$$

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We will consider the following central function on $\bigcup S_n$.

$$F_\omega(\sigma) = p_{t(\sigma)}(\omega),$$

where $t(\sigma)$ is the cycle-type of σ.

Examples

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$$\omega = ((1-q), (1-q)q, (1-q)q^2, \dots; 0, 0, \dots; 0)$$

$$\begin{aligned} F_\omega((1\ 2\ 3)(4\ 5)) &= \left(\sum_{i \geq 0} \left((1-q)q^i\right)^3 \right) \left(\sum_{i \geq 0} \left((1-q)q^i\right)^2 \right) \\ &= \frac{(1-q)^5}{(1-q^3)(1-q^2)} \\ &= F_\omega((1\ 2\ 3)(4\ 5)(6)) \end{aligned}$$

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$$\omega = (0, 0, \dots; 0, 0, \dots; 1)$$

$$F_\omega(\sigma) = \begin{cases} 1 \text{ if } \sigma = \mathrm{Id}; \\ 0 \text{ else.} \end{cases}$$

Probability measure on $\mathcal{Y}_n$

Fix a parameter ω an integer n.
F_ω / S_n is a central function on S_n. Therefore, as function on S_n,

$$F_\omega = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda.$$

Note that

$$\sum_{\lambda \vdash n} c_\lambda = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda(\mathrm{Id}) = F_\omega(\mathrm{Id}) = 1.$$

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Proposition

$$\forall\ \lambda, \quad c_\lambda \geq 0.$$

Hence, the c_λ define a probability measure $\mathbb{P}_n^\omega$ on $\mathcal{Y}_n$

$$\mathbb{P}_n^\omega(X = \lambda) = c_\lambda.$$

Pursuing the examples

- $\omega = (0, 0, \dots; 0, 0, \dots; 1)$.
  A classical result in representation theory states that:

$$\mathbb{C}[S_n] \simeq_{S_n} \bigoplus_{\lambda \vdash n} V_\lambda^{\dim(V_\lambda)}$$

We look at the trace of the action of σ:

$$F_\omega = \sum_{\lambda \vdash n} \frac{\dim(V_\lambda)^2}{n!} \chi_\lambda.$$

Hence $\mathbb{P}_n^\omega(X = \lambda) = \frac{\dim(V_\lambda)^2}{n!}$. This measure is known as Plancherel measure.

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- $\omega = ((1-q), (1-q)q, (1-q)q^2, \dots; 0, 0, \dots; 0)$.
  $\mathbb{P}_n^\omega$ is in this case a q-deformation of Plancherel measure already considered by Kerov in another context.

Main theorem of the talk

Theorem

Fix a parameter ω. For each n, we pick a random Young diagram $\lambda^{(n)}$ with the distribution $\mathbb{P}_{n,\omega}$. Then, one has the convergences in probability:

$$\forall\ i, \quad \frac{\lambda_i^{(n)}}{n} \to \alpha_i$$
$$\forall\ i, \quad \frac{(\lambda^{(n)})'_i}{n} \to \beta_i$$

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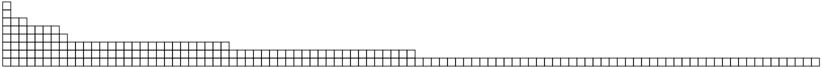
- proved by Kerov and Vershik in 1981.
- method of proof used here (which gives also information on the fluctuations): F., Méliot (2010) in the case of q-Plancherel measure.
- Generalization with the same argument to all $\mathbb{P}_n^\omega$: Méliot 2011.

Examples

- $q = 1/2, \ \omega = ((1-q), (1-q)q, (1-q)q^2, \dots; 0, 0, \dots; 0)$.

$$\frac{\lambda_i}{n} \to (1/2)^i, \quad \frac{\lambda'_i}{n} \to 0$$

Here is a random Young diagram of size 200 (computed and drawn by PL Méliot).



$$\lambda = (101, 51, 28, 8, 7, 3, 1, 1).$$

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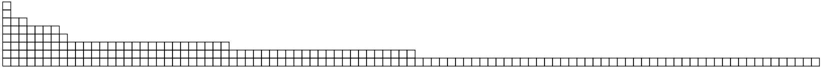

$$\lambda = (101, 51, 28, 8, 7, 3, 1, 1).$$

- $\omega = (0, 0 \dots; 0, 0, \dots; 1)$ (Plancherel case)

$$\frac{\lambda_i}{n} \to 0, \quad \frac{\lambda'_i}{n} \to 0$$

$\Rightarrow$ no big rows and columns (but no precise information!).

Normalized character values have simple expectations!

Fix $\sigma \in \mathfrak{S}_n$. Let us consider the random variable:

$$X_\sigma(\lambda) = \chi^\lambda(\sigma) = \frac{\mathrm{Tr}\left(\rho_\lambda(\sigma)\right)}{\dim V_\lambda}.$$

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Strategy: express other functions of Young diagrams in terms of X_σ.

Definition of normalized characters

Let us define

$$\mathrm{Ch}_\mu : \begin{array}{ccc} \mathcal{Y} & \to & \mathbb{C}; \\ \lambda & \mapsto & |\lambda|(|\lambda|-1)\dots(|\lambda|-k+1)\chi^\lambda(\sigma), \end{array}$$

where $k = |\mu|$.

and σ is a permutation in $S_{|\lambda|}$ of cycle type $\mu 1^{|\lambda|-k}$.

In particular,

$$\mathrm{Ch}_\mu(\lambda) = 0 \quad \text{as soon as } |\lambda| < |\mu|$$

Examples:

$$\mathrm{Ch}_{1^k}(\lambda) = |\lambda|(|\lambda|-1)\dots(|\lambda|-k+1)$$

$$\mathrm{Ch}_{(2)}(\lambda) = |\lambda|(|\lambda|-1)\chi^\lambda((1\ 2)) = \sum_i (\lambda_i)^2 - (\lambda'_i)^2$$

$$\mathrm{Ch}_{\mu\cup 1}(\lambda) = (|\lambda| - |\mu|)\,\mathrm{Ch}_\mu(\lambda)$$

Product of normalized characters

$$\mathrm{Ch}_\mu(\lambda) = |\lambda|(|\lambda|-1)\dots(|\lambda|-k+1)\chi^\lambda(\sigma).$$

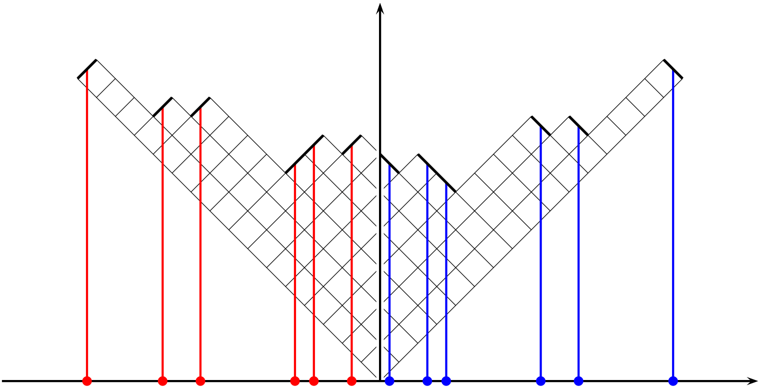
Proposition

The functions Ch_μ, when μ runs over all partitions, are linearly independent (over $\mathbb{C}$).
Moreover, they span a subalgebra $\Lambda^\star$ of functions on Young diagrams.

Example: $\mathrm{Ch}_{(2)} \cdot \mathrm{Ch}_{(2)} = 4 \cdot \mathrm{Ch}_{(3)} + \mathrm{Ch}_{(2,2)} + 2\,\mathrm{Ch}_{(1,1)}$.

Frobenius coordinates and their power sums

If λ is a Young diagram, define its Forbenius coordinates $(a_i, b_i), 1 \leq i \leq h$ as follows:



$$b_i = \lambda'_i - i + 1/2 > 0, \quad a_i = \lambda_i - i + 1/2 > 0$$

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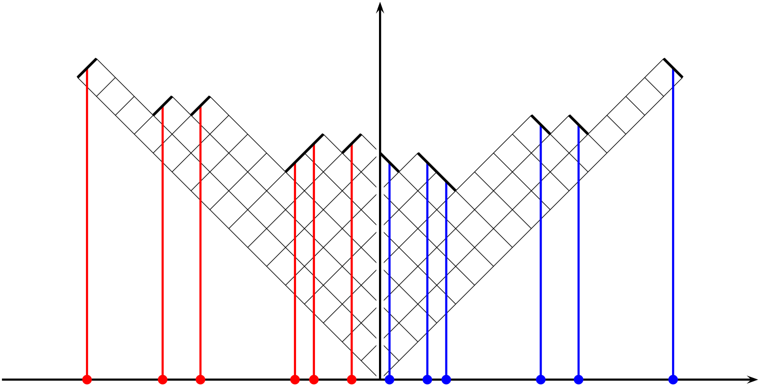

$$M_k(\lambda) := \sum a_i^k + (-1)^{k-1} \sum_i b_i^k$$

An algebraic basis of $\Lambda^\star$

Theorem (Kerov, Olshanski, 1994)

$(\lambda \mapsto M_k(\lambda))_{k \geq 1}$ is an algebraic basis of $\Lambda^\star$.

Example:

$$\mathrm{Ch}_4 = M_4 - 4M_2 \cdot M_1 + \frac{11}{2} M_2.$$

Not very explicit formula for the change of basis

$$\mathrm{Ch}_k = [t^{k+1}] \left\{ -\frac{1}{k} \prod_{j=1}^{k} (1 - (j - 1/2)t) \cdot \exp \left(\sum_{j=1}^{\infty} \frac{M_j\, t^j}{j} (1 - (1 - kt)^{-j}) \right) \right\}$$

Asymptotic change of basis

We consider a gradation on $\Lambda^\star$:

$$\deg(M_k) = k$$

The highest degree term of the change of basis is easy:

$$\mathrm{Ch}_\mu = \prod_i M_{\mu_i} + \text{ smaller degree terms.}$$

Filtration and order of magnitude

Lemma

$$x \in \Lambda^\star \Rightarrow \mathbb{E}_{\mathbb{P}_n^\omega}(x) = O(n^{\deg(x)})$$

Proof:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_n^\omega}(\mathrm{Ch}_\mu) &= n(n-1)\dots(n-|\mu|+1)E\big(\lambda \mapsto \chi^\lambda(\mu)\big)\\
&= n(n-1)\dots(n-|\mu|+1)p_\mu(\omega)\\
&= O(n^{|\mu|})
\end{aligned}$$

so the lemma is true for $x = \mathrm{Ch}_\mu$.
As it is a basis, this is enough. $\square$

Asymptotic behaviour of the M_k

Using the previous lemma,

$$\mathbb{E}_{\mathbb{P}_n^\omega}(M_k) = \mathbb{E}_{\mathbb{P}_n^\omega}(\mathrm{Ch}_{(k)}) + O(n^{k-1})$$

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We can also estimate its variance:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_n^\omega}(M_k^2) - \mathbb{E}_{\mathbb{P}_n^\omega}(M_k)^2 &= \mathbb{E}_{\mathbb{P}_n^\omega}(\mathrm{Ch}_{(k,k)}) - \mathbb{E}_{\mathbb{P}_n^\omega}(\mathrm{Ch}_{(k)})^2 \\ &\qquad + O(n^{2k-1}) \\ &= n^{2k} p_{(k,k)}(\omega) - (n^k p_k(\omega))^2 + O(n^{2k-1}) \\ &= O(n^{2k-1})\end{aligned}$$

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$\dfrac{M_k(\lambda)}{n^k}$ converges in probability towards $p_k(\omega)$.

End of the proof of the theorem

$\frac{M_k(\lambda)}{n^m}$ is the $(k-1)$-th moment of the probability measure

$$X_\lambda = \sum_{i=1}^{d} (a_i^*(\lambda)/n)\, \delta_{(a_i^*(\lambda)/n)} + (b_i^*(\lambda)/n)\, \delta_{(-b_i^*(\lambda)/n)}.$$

and $p_k(omega)$ the $k-1$-th moment of

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$$X_\omega = \gamma\delta_0 + \sum_{i\geq 1} \alpha_i \delta_{\alpha_i} + \beta_i \delta_{-\beta_i}.$$

We have convergence in probability of the repartition function at each point $x \neq 0, \alpha_i, -\beta_i$.

$\Rightarrow$ easy to check that it implies the theorem.

Fluctuations of the M_k

- it is easy to compute the dominant term of $\mathrm{Var}(M_k)$ and $\mathrm{Cov}(M_k, M_l)$.
- One can show that the fluctuations of the M_k's form a Gaussian vector (using joint cumulants).
  It requires combinatorial computations in $\Lambda^\star$ and some technical tools (introduced by P. Śniady)

Fluctuations of row and column lengths

Consider the case $\beta_1 = \beta_2 = \dots = \gamma = 0$. Let

$$Y_i = \sqrt{n} \left(\frac{\lambda_i}{n} - \alpha_i \right).$$

Then

$$M_k(\lambda) \sim \sum_i \lambda_i^k = n^k \left(\sum \alpha_i^k + \frac{k}{\sqrt{n}} \sum \alpha_i^{k-1} Y_i + \dots \right)$$

i.e.

$$\forall\ k, \quad k \sum \alpha_i^{k-1} Y_i = \text{fluctuations}(M_k).$$

We can recover the Y_i only if the α_i are distinct!

Fluctuations of row and column lengths (2)

- If the α_i are distinct, it's working. For instance, if $\omega = ((1-q), (1-q)q, (1-q)q^2, \dots; 0, 0, \dots; 0)$,

Theorem (F., Méliot 2010)

Denote $Y_{n,q,i}$ the rescaled deviation

$$\sqrt{n}\left(\frac{\lambda_i}{n} - (1-q)\,q^{i-1}\right).$$

Then we have convergence of the finite-dimensional laws of the random process $(Y_{n,q,i})_{i\geq 1}$ towards those of a gaussian process $(Y_{q,i})_{i\geq 1}$ with:

$$\mathbb{E}[Y_{q,i}] = 0 \quad ; \quad \mathbb{E}[Y_{q,i}^2] = (1-q)\,q^{i-1} - (1-q)^2\,q^{2(i-1)} \quad ;$$
$$\operatorname{cov}(Y_{q,i}, Y_{q,j}) = -(1-q)^2\,q^{i+j-2}.$$

- Otherwise, we need another method (cf. Pierre-Loïc's talk).

Remark on the Plancherel case

- The theorem does not give much information.
- There is a deterministic limit shape (Logan, Shepp 77 and Kerov, Vershik 77). The fluctuations around this shape is known (Ivanov, Kerov, Olshanski 2002).
- They used the same kind of method, replacing power sums of Frobenius coordinates by free cumulant of the transition measure (in fact, we have adapted their ideas!).

End of the talk. Thanks!